**UNIVERSITÄT DES SAARLANDES**
**DEPARTMENT 6.1 – MATHEMATICS**
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**Exercises for the Lecture**
**Differential Geometry**
Summer Term 2020

**Sheet 1** **Submission:** /

**Resources: Lessons 1 – 3; Sections 1-1 – 1-4 in [Car16]**

**Exercise 1.**
For all $t \in \mathbb{R}$, the straight line through $(0,1)$ and $(t,0)$ cuts the unit circle $K = \{(x,y) \in \mathbb{R}^2 \, ; \, x^2+y^2=1\}$ in exactly one point which is different to $(0,1)$ and which will be denoted by $(x(t),y(t))$.

(i) Determine the functions $x, y\colon \mathbb{R} \to \mathbb{R}$ and show that $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (x(t),y(t))$ is a regular parametrization of $K \setminus \{(0,1)\}$.

(ii) Calculate the arc length of the curve $\alpha|_{[-1,1]}\colon [-1,1] \to \mathbb{R}^2,\ t \mapsto \alpha(t)$.

**Exercise 2.**
Justify that the following curves in $\mathbb{R}^3$ have finite arc lengths and calculate them:

(i) $\beta\colon [0,1] \to \mathbb{R}^3,\ t \mapsto (6t, 3t^2, t^3)$,

(ii) $\gamma\colon [0,\sqrt{2}] \to \mathbb{R}^3,\ t \mapsto (t, t\sin(t), t\cos(t))$.

(*Hint: You can use the following identity without proving it:* $\int_0^s \sqrt{1+t^2}\,\mathrm{d}t = \frac{1}{2}(\sqrt{1+s^2}\cdot s + \operatorname{arsinh}(s))$ *with* $s > 0$.)

**Exercise 3.**
Reparameterize the following curves by arc length: (*Hint: Remark 2 on p. 23 in [Car16] can be useful.*)

(i) $\delta\colon (1,\infty) \to \mathbb{R}^3,\ t \mapsto e^{-t}(\cos(t), \sin(t), 1)$,

(ii) $\varepsilon\colon (0,\infty) \to \mathbb{R}^3,\ t \mapsto (e^t, e^{-t}, \sqrt{2}t)$.

**Exercise 4.**
Let $\alpha\colon I \to \mathbb{R}^3$ ($I \subset \mathbb{R}$ an interval) be a regular curve, $[a,b] \subset I$ and $A = \alpha(a)$, $B = \alpha(b)$ with $A \neq B$. Show:

(i) For each unit vector $e \in \mathbb{R}^3$, we have

$$(B - A)\cdot e \leq L_\alpha,$$

where $L_\alpha$ is the arc length between $A$ and $B$ with respect to $\alpha$.

(ii) The shortest arc length of any curve connecting $A$ and $B$ is the straight line connecting them.

# References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.