UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 10 **Submission:** /

Resources: Up to Lesson 17; Up to p. 73 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – 3-3 in [Car16]

Exercise 1.
Let $\Omega \subset \mathbb{R}^2$ be a domain, $w \in \Omega$, and let $X\colon \Omega \to \mathbb{R}^3$ be a parameterized surface. Show: The mapping

$$III_w\colon T_wX \times T_wX \to \mathbb{R},\ (U,V) \mapsto S_w(U)\cdot S_w(V)$$

is a symmetric bilinear form (*third fundamental form of* X) and the following relationship holds:

$$III_w - (\kappa_1(w) + \kappa_2(w))II_w + \kappa_1(w)\kappa_2(w)I_w \equiv 0.$$

Here $\kappa_{1,2}(w)$ are the principal curvatures of X at w.

Exercise 2.
Show the *parameter invariance of the area*: Let $\Omega, \tilde\Omega$ be two domains, $X\colon \Omega \to \mathbb{R}^3$ a regular parameterized surface, $\varphi\colon \tilde\Omega \to \Omega$ a diffeomorphism and $\tilde X = X \circ \varphi$. Then

$$\int_\Omega |X_u(u,v) \times X_v(u,v)|\,\mathrm{d}u\,\mathrm{d}v = \int_{\tilde\Omega} |\tilde X_{\tilde u}(\tilde u,\tilde v) \times \tilde X_{\tilde v}(\tilde u,\tilde v)|\,\mathrm{d}\tilde u\,\mathrm{d}\tilde v.$$

Exercise 3.
Let $\Omega \subset \mathbb{R}^2$ be a domain and let $X\colon \Omega \to \mathbb{R}^3$ be a twice continuous differentiable parameterized surface. Define

$$X^\varepsilon\colon \Omega \to \mathbb{R}^3,\ (u,v) \mapsto X(u,v) + \varepsilon\varphi(u,v)N(u,v),$$

where $\varepsilon \in \mathbb{R}$, $\varphi \in C_c^\infty(\Omega)$ and $N(u,v)$ is the normal vector of X in (u,v). Show: For a sufficiently small $\varepsilon_0 > 0$, X^ε is a regular parameterized surface for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. We call this a *normal variation of* X.

(*Hint: It can be useful to first show the formula in Exercise 4 (i).*)

Exercise 4.
Let $\Omega \subset \mathbb{R}^2$ be a domain. A *minimal surface* is a parameterized surface $X\colon \Omega \to \mathbb{R}^3$ with vanishing mean curvature $H \equiv 0$. In this exercise we want show that a minimal surface can also be characterzied via the extremality (minimum *or* maximum!) of the area with respect to all its normal variations (see Exercise 3).

(i) Let X^ε be a normal variation of X (see Exercise 3). Show that for the coefficients $\mathcal{E}^\varepsilon, \mathcal{F}^\varepsilon$ and $\mathcal{G}^\varepsilon$of the first fundamental form of X^ε the following relationship holds:

$$\mathcal{E}^\varepsilon\mathcal{G}^\varepsilon - (\mathcal{F}^\varepsilon)^2 = (\mathcal{E}^0\mathcal{G}^0 - (\mathcal{F}^0)^2)(1 - 4\varepsilon\varphi H) + R,$$

where $R = R(u,v,\varepsilon)$ with $\lim_{\varepsilon\to 0} R(u,v,\varepsilon)/\varepsilon = 0$.

(ii) Conclude that for the area $\mathcal{A}(\varepsilon) = A(X^\varepsilon)$ the following holds:

$$\mathcal{A}'(0) = 0 \quad\Longleftrightarrow\quad H \equiv 0,$$

i.e. $\varepsilon = 0$ is a stationary point of the area functional and the area has an extremum for X if and only if the mean curvature vanishes.

(please turn the page)

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.