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DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 10, Solution **Submission:** /

Resources: Up to Lesson 17; Up to p. 73 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – 3-3 in [Car16]

Exercise 1.
Let $\Omega \subset \mathbb{R}^2$ be a domain, $w \in \Omega$, and let $X\colon \Omega \to \mathbb{R}^3$ be a parameterized surface. Show: The mapping

$$III_w\colon T_wX \times T_wX \to \mathbb{R},\ (U,V) \mapsto S_w(U) \cdot S_w(V)$$

is a symmetric bilinear form (*third fundamental form of* X) and the following relationship holds:

$$III_w - (\kappa_1(w) + \kappa_2(w))II_w + \kappa_1(w)\kappa_2(w)I_w \equiv 0.$$

Here $\kappa_{1,2}(w)$ are the principal curvatures of X at w.

Solution 1.
The bilinearity and symmetry are clear.

Let V_1 and V_2 be the principal directions. Then

$$\begin{aligned} &III_w(V_i,V_j) - (\kappa_1(w)+\kappa_2(w))II_w(V_i,V_j) + \kappa_1(w)\kappa_2(w)I_w(V_i,V_j) \\ =&(\kappa_i(w)\kappa_j(w) - (\kappa_1(w)+\kappa_2(w))\kappa_i(w) + \kappa_1(w)\kappa_2(w))\langle V_i, V_j\rangle \\ =&0 \end{aligned}$$

for all $i,j \in \{1,2\}$. Since (V_1,V_2) is an orthonormal basis of T_wX, the result follows.

Exercise 2.
Show the *parameter invariance of the area*: Let $\Omega, \tilde{\Omega}$ be two domains, $X\colon \Omega \to \mathbb{R}^3$ a regular parameterized surface, $\varphi\colon \tilde{\Omega} \to \Omega$ a diffeomorphism and $\tilde{X} = X \circ \varphi$. Then

$$\int_\Omega |X_u(u,v) \times X_v(u,v)| \,\mathrm{d}u\,\mathrm{d}v = \int_{\tilde{\Omega}} |\tilde{X}_{\tilde{u}}(\tilde{u},\tilde{v}) \times \tilde{X}_{\tilde{v}}(\tilde{u},\tilde{v})| \,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}.$$

Solution 2.
Let $(\tilde{u},\tilde{v}) \in \tilde{\Omega}$. We have

$$\tilde{G}_{(\tilde{u},\tilde{v})} = D\tilde{X}^T_{(\tilde{u},\tilde{v})} D\tilde{X}_{(\tilde{u},\tilde{v})} = D\varphi^T_{(\tilde{u},\tilde{v})} DX^T_{\varphi(\tilde{u},\tilde{v})} DX_{\varphi(\tilde{u},\tilde{v})} D\varphi_{(\tilde{u},\tilde{v})},$$

hence

$$\det(\tilde{G}_{(\tilde{u},\tilde{v})}) = \det(DX^T_{\varphi(\tilde{u},\tilde{v})} DX_{\varphi(\tilde{u},\tilde{v})}) \det(D\varphi_{(\tilde{u},\tilde{v})})^2.$$

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With a change of variables we obtain

$$\begin{aligned}\int_{\tilde{\Omega}} |\partial_1 \tilde{X} \times \partial_2 \tilde{X}| \,\mathrm{d}\lambda &= \int_{\tilde{\Omega}} \sqrt{\det(\tilde{G}_{(\tilde{u},\tilde{v})})} \,\mathrm{d}\lambda(\tilde{u},\tilde{v}) \\ &= \int_{\tilde{\Omega}} \sqrt{\det(DX^T_{\varphi(\tilde{u},\tilde{v})} DX_{\varphi(\tilde{u},\tilde{v})})} |\det(D\varphi_{(\tilde{u},\tilde{v})})| \,\mathrm{d}\lambda(\tilde{u},\tilde{v}) \\ &= \int_{\Omega} \sqrt{\det(DX^T_{(u,v)} DX_{(u,v)})} \,\mathrm{d}\lambda(u,v) \\ &= \int_{\Omega} \sqrt{\det(G_{(u,v)})} \,\mathrm{d}\lambda(u,v) \\ &= \int_{\Omega} |\partial_1 X \times \partial_2 X| \,\mathrm{d}\lambda.\end{aligned}$$

Exercise 3.
Let $\Omega \subset \mathbb{R}^2$ be a domain and let $X\colon \Omega \to \mathbb{R}^3$ be a twice continuous differentiable parameterized surface. Define

$$X^\varepsilon\colon \Omega \to \mathbb{R}^3,\ (u,v) \mapsto X(u,v) + \varepsilon\varphi(u,v)N(u,v),$$

where $\varepsilon \in \mathbb{R}$, $\varphi \in C_c^\infty(\Omega)$ and $N(u,v)$ is the normal vector of X in (u,v). Show: For a sufficiently small $\varepsilon_0 > 0$, X^ε is a regular parameterized surface for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. We call this a *normal variation of* X.

(*Hint: It can be useful to first show the formula in Exercise 4 (i).*)

Solution 3.
We first show the formula in Exercise 4 (i): We have

$$\partial_i X^\varepsilon = \partial_i X + \varepsilon(\partial_i\varphi N + \varphi\partial_i N) = \partial_i X + \varepsilon R_i$$

with $R_i = \partial_i\varphi N + \varphi\partial_i N \in C_c(\Omega, \mathbb{R}^3)$ for $i = 1, 2$. Furthermore, we see that

$$\langle \partial_i X, R_j \rangle = -\varphi\langle X, \partial_{ij} N\rangle$$

for $i, j = 1, 2$, hence

$$\begin{aligned}\mathcal{E}^\varepsilon &= \mathcal{E}^\varepsilon - 2\varepsilon\varphi\mathcal{L} + \varepsilon^2\|R_1\|^2, \\ \mathcal{G}^\varepsilon &= \mathcal{G}^\varepsilon - 2\varepsilon\varphi\mathcal{N} + \varepsilon^2\|R_2\|^2, \\ \mathcal{F}^\varepsilon &= \mathcal{F}^\varepsilon - 2\varepsilon\varphi\mathcal{M} + \varepsilon^2\langle R_1, R_2\rangle.\end{aligned}$$

Therefore we obtain with the identity on p. 71 in [Fuc08] (or equation (5) on p. 158 in [Car16])

$$\det(G^\varepsilon) = \mathcal{E}^\varepsilon\mathcal{G}^\varepsilon - (\mathcal{F}^\varepsilon)^2 = \det(G)(1 - 4\varepsilon\varphi H) + R = \det(G)\left(1 - 4\varepsilon\varphi H + \frac{R}{\det(G)}\right),$$

where $R \in C_c(\Omega)$ and $R \in \mathcal{O}(\varepsilon^2)$. Since

$$\lim_{\varepsilon\to 0} -4\varepsilon\varphi H + \frac{R}{\det(G)} = 0$$

and $R/\det(G) \in C_c(\Omega)$, the result follows with $\det(G) > 0$.

(*Hint: For the derivation also see p. 82ff in [Fuc08] or p. 200f in [Car16].*)

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Exercise 4.
Let $\Omega \subset \mathbb{R}^2$ be a domain. A *minimal surface* is a parameterized surface $X\colon \Omega \to \mathbb{R}^3$ with vanishing mean curvature $H \equiv 0$. In this exercise we want show that a minimal surface can also be characterzied via the extremality (minimum *or* maximum!) of the area with respect to all its normal variations (see Exercise 3).

(i) Let X^ε be a normal variation of X (see Exercise 3). Show that for the coefficients $\mathcal{E}^\varepsilon, \mathcal{F}^\varepsilon$ and $\mathcal{G}^\varepsilon$of the first fundamental form of X^ε the following relationship holds:

$$\mathcal{E}^\varepsilon \mathcal{G}^\varepsilon - (\mathcal{F}^\varepsilon)^2 = (\mathcal{E}^0 \mathcal{G}^0 - (\mathcal{F}^0)^2)(1 - 4\varepsilon\varphi H) + R,$$

where $R = R(u, v, \varepsilon)$ with $\lim_{\varepsilon\to 0} R(u, v, \varepsilon)/\varepsilon = 0$.

(ii) Conclude that for the area $\mathcal{A}(\varepsilon) = A(X^\varepsilon)$ the following holds:

$$\mathcal{A}'(0) = 0 \quad \iff \quad H \equiv 0,$$

i.e. $\varepsilon = 0$ is a stationary point of the area functional and the area has an extremum for X if and only if the mean curvature vanishes.

Solution 4.

(i) See Exercise 3.

(ii) Let $\varepsilon_0 > 0$ be from Exercise 3. Define

$$f\colon (-\varepsilon_0, \varepsilon_0) \times \Omega \to \mathbb{R},\ (\varepsilon, (u, v)) \mapsto \sqrt{\det(G^\varepsilon_{(u,v)})}$$

We obtain

$$\mathcal{A}'(\varepsilon) = \partial_1 \int_\Omega f(\varepsilon, \cdot)\,\mathrm{d}\lambda = \int_\Omega \partial_1 f(\varepsilon, \cdot)\,\mathrm{d}\lambda = \int_\Omega \frac{-4\varphi H \det(G) + \partial_1 R(\varepsilon, \cdot)}{2\sqrt{\det(G^\varepsilon)}}\,\mathrm{d}\lambda,$$

hence

$$\mathcal{A}'(0) = -2 \int_\Omega \varphi H \sqrt{\det(G)}\,\mathrm{d}\lambda = -2\langle \varphi, H\sqrt{\det(G)}\rangle_{L^2(\Omega)}.$$

Since this formula holds for all test functions φ and the test functions are dense in $L^2(\Omega)$, the result follows from the regularity of X.

(*Hint: Also see p. 82f in [Fuc08] or p. 200f in [Car16].*)

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.