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DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 11, Solution **Submission:** /

Resources: Up to Lesson 20; Chapters 1–2 in [Fuc08]; Chapters 1–3 in [Car16]

Exercise 1.

(i) Show that the parametrization of the elliptic paraboloid

$$X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,u^2+v^2)$$

has no asymptotic curves.

(ii) Determine the asymptotic curves of the following parametrization of the hyperbolic paraboloid

$$X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,u^2-v^2).$$

Solution 1.

(i) Let $f\colon \mathbb{R}^2 \to \mathbb{R},\ (u,v) \mapsto u^2+v^2$. By p. 73 in [Fuc08] (or p. 166 in [Car16]), we have

$$K(u,v) = \frac{4}{(1+4u^2+4v^2)^2} > 0$$

for all $(u,v) \in \mathbb{R}^2$. Hence by p. 74 in [Fuc08] it follows that there exists no asymptotic curves on X.

(ii) Let $f\colon \mathbb{R}^2 \to \mathbb{R},\ (u,v) \mapsto u^2-v^2$, hence $X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,f(u,v))$. Let $\omega\colon I \to \mathbb{R}^2$ be a smooth curve, $\gamma = X \circ \omega$ and $(u,v) \in \mathbb{R}^2$. A simple calculation (or using p. 49 in [Fuc08]) shows that

$$N(u,v) = \frac{1}{\sqrt{1+4u^2+4v^2}}(-2u,2v,1)$$

and

$$\mathcal{L}(u,v) = \frac{2}{\sqrt{1+4u^2+4v^2}},$$
$$\mathcal{N}(u,v) = -\frac{2}{\sqrt{1+4u^2+4v^2}},$$
$$\mathcal{M}(u,v) = 0.$$

With (6) on p. 74 in [Fuc08] (or (7) on p. 162 in [Car16]) it follows that

$$\kappa_n = 0 \iff (\omega_1')^2 = (\omega_2')^2,$$

thus

$$\omega_1' = \pm\omega_2'$$

and therefore

$$\omega_1 = \pm\omega_2 + C \qquad (C \in \mathbb{R}).$$

The asymptotic curves are $\gamma_{C,\pm}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (\pm t + C, t, (\pm t + C)^2 - t^2)$ for all $C \in \mathbb{R}$.

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Exercise 2.

(See Exercise 2 in Section 3-3 in [Car16].)

Let $a, b > 0$. Consider the *helicloid*

$$X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (av\cos(u), av\sin(u), bu).$$

(i) Show that X is a ruled surface. Are the generators asymptotic curves?

(ii) Determine the curvature lines of the surface for $a = b = 1$.

(Hint: Use $\widetilde{\omega_2} = \operatorname{arsinh}(\omega_2)$.)

(iii) Show that X is a minimal surface.

Solution 2.

(i) We have

$$X(u,v) = (0,0,bu) + v(a\cos(u), a\sin(u), 0) = \alpha(u) + vw(u)$$

for all $(u,v) \in \mathbb{R}^2$ with

$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (0,0,bt) \quad \text{and} \quad w\colon \mathbb{R} \to \mathbb{R}^3 \setminus \{0\},\ t \mapsto (a\cos(t), a\sin(t), 0),$$

hence X is a ruled surface. For all $(u,v) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\partial_1 X(u,v) &= \alpha'(u) + vw'(u) = (-av\sin(u), av\cos(u), b),\\
\partial_2 X(u,v) &= w(u) = a(\cos(u), \sin(u), 0),\\
\partial_1 X(u,v) \times \partial_2 X(u,v) &= (-ab\sin(u), ab\cos(u), -va^2),\\
\partial_{11} X(u,v) &= vw''(u) = -va(\cos(u), \sin(u), 0),\\
\partial_{12} X(u,v) &= w'(u) = a(-\sin(u), \cos(u), 0),\\
\partial_{22} X(u,v) &= (0,0,0).
\end{aligned}$$

Thus we obtain that

$$I_{(u,v)} = \begin{pmatrix} a^2v^2 + b^2 & 0 \\ 0 & a^2 \end{pmatrix} \quad \text{and} \quad II_{(u,v)} = \frac{ab}{\sqrt{b^2 + a^2v^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $(u,v) \in \mathbb{R}^2$. Fix $u \in \mathbb{R}$ and set $\omega\colon \mathbb{R} \to \mathbb{R}^2$, $v \mapsto (u,v)$, hence $\gamma = X \circ \omega$ is the generator with respect to u. It follows that

$$\left\langle II_{(u,v)} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \right\rangle = \frac{ab}{\sqrt{b^2 + a^2v^2}} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0$$

for all $v \in \mathbb{R}$, thus the generators are asymptotic curves.

(ii) By p. 75 in [Fuc08] (or p. 163 in [Car16]), we have to solve the following differential equation:

$$\begin{aligned}
0 &= (\mathcal{E}\mathcal{M} - \mathcal{F}\mathcal{L})\omega_1'^2 + (\mathcal{E}\mathcal{N} - \mathcal{G}\mathcal{L})\omega_1'\omega_2' + (\mathcal{F}\mathcal{N} - \mathcal{G}\mathcal{M})\omega_2'^2\\
&= \frac{ab}{\sqrt{b^2 + a^2\omega_2^2}}((a^2\omega_2^2 + b^2)\omega_1'^2 - a^2\omega_2'^2)\\
&= \frac{a^3b}{\sqrt{b^2 + a^2\omega_2^2}}\left(\left(\omega_2^2 + \left(\frac{b}{a}\right)^2\right)\omega_1'^2 - \omega_2'^2\right)
\end{aligned}$$

resp. with $a = b = 1$

$$0 = \frac{1}{\sqrt{1 + \omega_2^2}}((\omega_2^2 + 1)\omega_1'^2 - \omega_2'^2) \iff \omega_1'^2 = \frac{1}{\omega_2^2 + 1}\omega_2'^2,$$

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where $\omega\colon \mathbb{R} \to \mathbb{R}^2$ is a smooth function. Set $\tilde{\omega}\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\omega_1(t), \operatorname{arsinh}(\omega_2(t)))$, hence

$$\tilde{\omega_1}'^2 = \omega_1'^2 = \frac{1}{\omega_2^2+1}\omega_2'^2 = \frac{1}{\cosh(\tilde{\omega_2})^2}\cosh(\tilde{\omega_2})^2\tilde{\omega_2}'^2 = \tilde{\omega_2}'^2.$$

Thus

$$\tilde{\omega_1} = \pm\tilde{\omega_2} + C \quad \text{resp.} \quad \omega_1 = \pm \operatorname{arsinh}(\omega_2) + C \qquad (C \in \mathbb{R}).$$

Therefore the curvature lines are

$$\gamma_{\pm,C}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} a\omega_2(t)\cos(\pm \operatorname{arsinh}(\omega_2(t)) + C) \\ a\omega_2(t)\sin(\pm \operatorname{arsinh}(\omega_2(t)) + C) \\ b(\pm \operatorname{arsinh}(\omega_2(t)) + C) \end{pmatrix} \quad (C \in \mathbb{R}).$$

With, e.g. $\omega_2 = \mathrm{id}$, it follows that

$$\gamma_{\pm,C}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} at\cos(\pm \operatorname{arsinh}(t) + C) \\ at\sin(\pm \operatorname{arsinh}(t) + C) \\ b(\pm \operatorname{arsinh}(t) + C) \end{pmatrix} \quad (C \in \mathbb{R}).$$

(iii) By p. 71 in [Fuc08] (or (5) on p. 158 in [Car16]), we have

$$H = \frac{1}{2}\frac{1}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}(\mathcal{L}\mathcal{G} + \mathcal{E}\mathcal{N} - 2\mathcal{F}\mathcal{M}) = \frac{1}{2}\frac{1}{a^2(a^2v^2+b^2)}(0+0-0) = 0,$$

hence X is a minimal surface.

Exercise 3.

(See Exercise 6 in Section 3-3 in [Car16].)

(i) Let the unit sphere be parameterized by

$$X\colon (0,2\pi)\times(0,\pi) \to \mathbb{R}^3,\ (u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)).$$

Calculate the geodesic curvature of all circles of latitude and longitude (u resp. v coordinate lines).

(ii) The *pseudo sphere* is the following regular parameterized rotation surface

$$P^2\colon \mathbb{R}\setminus\{0\}\times\mathbb{R} \to \mathbb{R}^3,\ (u,v) \mapsto \left(\frac{\cos(v)}{\cosh(u)}, \frac{\sin(v)}{\cosh(u)}, u - \tanh(u)\right).$$

Show that the pseudo sphere has constant negative Gauß curvature.

Solution 3.

(i) We first observe that

$$N(u,v) = -(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

for all $(u,v) \in (0,2\pi)\times(0,\pi)$.

Fix $u \in (0,2\pi)$. Let

$$\alpha\colon (0,\pi) \to \mathbb{R}^3,\ v \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)).$$

The arc length is given by

$$s(v) = \int_0^v |\alpha'(\tau)|\,\mathrm{d}\tau = \int_0^v 1\,\mathrm{d}\tau = v$$

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for all $v \in (0, \pi)$, hence α is parameterized by arc length. Then

$$\begin{aligned} t(s) &= \alpha'(s) = (\cos(u)\cos(s), \sin(u)\cos(s), -\sin(s)), \\ t'(s) &= -\alpha(s) = -(\cos(u)\sin(s), \sin(u)\sin(s), \cos(s)), \\ \overline{s}(s) &= \overline{N}(s) \times t(s) = N(u,s) \times t(s) = (\sin(u), -\cos(u), 0) \end{aligned}$$

for all $s \in (0, \pi)$, thus

$$\kappa_g(s) = \langle t'(s), \overline{s}(s) \rangle = 0$$

for all $s \in (0, \pi)$, i.e. the circle of longitude are geodesics.

Now fix $v \in (0, \pi)$. Let

$$\tilde{\alpha}\colon (0, 2\pi) \to \mathbb{R}^3,\ u \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)).$$

The arc length is given by

$$s(u) = \int_0^u |\tilde{\alpha}'(\tau)| \,\mathrm{d}\tau = \int_0^u \sin(v) \,\mathrm{d}\tau = \sin(v)u$$

for all $u \in (0, 2\pi)$, thus

$$\alpha\colon (0, 2\pi\sin(v)) \to \mathbb{R}^3,\ s \mapsto \left(\cos\left(\frac{s}{\sin(v)}\right)\sin(v), \sin\left(\frac{s}{\sin(v)}\right)\sin(v), \cos(v)\right)$$

is the reparametrization of $\tilde{\alpha}$ by arc length. Then

$$\begin{aligned} t(s) &= \alpha'(s) = \left(-\sin\left(\frac{s}{\sin(v)}\right), \cos\left(\frac{s}{\sin(v)}\right), 0\right), \\ t'(s) &= -\left(\cos\left(\frac{s}{\sin(v)}\right)\frac{1}{\sin(v)}, \sin\left(\frac{s}{\sin(v)}\right)\frac{1}{\sin(v)}, 0\right), \\ \overline{s}(s) &= \overline{N}(s) \times t(s) \\ &= N\left(\frac{s}{\sin(v)}, v\right) \times t(s) = \left(\cos\left(\frac{s}{\sin(v)}\right)\cos(v), \sin\left(\frac{s}{\sin(v)}\right)\cos(v), -\sin(v)\right) \end{aligned}$$

for all $s \in (0, 2\pi\sin(v))$, hence

$$\kappa_g(s) = \langle t'(s), \overline{s}(s) \rangle = -\frac{\cos(v)}{\sin(v)}$$

for all $s \in (0, 2\pi\sin(v))$. (Therefore, the equator ($v = \pi/2$) is the only geodesic.)

(ii) Let $(u, v) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$. We have

$$\begin{aligned} \partial_1 P^2(u,v) &= \frac{\sinh(u)}{\cosh(u)^2}(-\cos(v), -\sin(v), \sinh(u)), \\ \partial_2 P^2(u,v) &= \frac{1}{\cosh(u)}(-\sin(v), \cos(v), 0), \\ \partial_1 P^2(u,v) \times \partial_2 P^2(u,v) &= -\frac{\sinh(u)}{\cosh(u)^3}(\sinh(u)\cos(v), \sinh(u)\sin(v), 1), \\ |\partial_1 P^2(u,v) \times \partial_2 P^2(u,v)| &= \frac{|\sinh(u)|}{\cosh(u)^2}, \\ N(u,v) &= -\frac{\sinh(u)}{|\sinh(u)|}\frac{1}{\cosh(u)}(\sinh(u)\cos(v), \sinh(u)\sin(v), 1), \\ \partial_{11} P^2(u,v) &= -\frac{1}{\cosh(u)^3}((1-\sinh(u)^2)\cos(v), (1-\sinh(u)^2)\sin(v), -2\sinh(u)), \\ \partial_{12} P^2(u,v) &= \frac{\sinh(u)}{\cosh(u)^2}(\sin(v), -\cos(v), 0), \\ \partial_{22} P^2(u,v) &= -\frac{1}{\cosh(u)}(\cos(v), \sin(v), 0), \end{aligned}$$

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hence

$$I_{(u,v)} = \tanh(u)^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sinh(u)^2} \end{pmatrix} \quad \text{and} \quad II_{(u,v)} = \frac{1}{|\sinh(u)|} \tanh(u)^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With p. 71 in [Fuc08] (or (4) on p. 158 in [Car16]) we obtain

$$K(u,v) = \frac{\det(II_{(u,v)})}{\det(I_{(u,v)})} = -1.$$

Exercise 4.
Let $\Omega \subset \mathbb{R}^2$ be a domain and let $X \colon \Omega \to \mathbb{R}^3$ be a regular parametrization of a surface. Show that the following statements are equivalent:

(i) $H \equiv K \equiv 0$ on Ω.

(ii) $X(\Omega)$ is a subset of a plane.

Solution 4.
Let $H \equiv K \equiv 0$ on Ω. Then $\kappa_1 = \kappa_2 = 0$ and hence with Satz 10 on p. 67 in [Fuc08] (or Proposition 4 on p. 149 in [Car16]) the implication follows.

If $X(\Omega)$ is a subset of a plane, then $DN = 0$ and hence $\kappa_1 = \kappa_2 = 0$. Therefore, $H \equiv K \equiv 0$ on Ω.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.