UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 1, Solution **Submission:** /

Resources: Lessons 1 – 3; Sections 1-1 – 1-4 in [Car16]

Exercise 1.
For all $t \in \mathbb{R}$, the straight line through $(0,1)$ and $(t,0)$ cuts the unit circle $K = \{(x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2=1\}$ in exactly one point which is different to $(0,1)$ and which will be denoted by $(x(t),y(t))$.

(i) Determine the functions $x,y\colon \mathbb{R}\to\mathbb{R}$ and show that $\alpha\colon \mathbb{R}\to\mathbb{R}^2,\ t\mapsto (x(t),y(t))$ is a regular parametrization of $K\setminus\{(0,1)\}$.

(ii) Calculate the arc length of the curve $\alpha|_{[-1,1]}\colon [-1,1]\to\mathbb{R}^2,\ t\mapsto\alpha(t)$.

Solution 1.

(i) Let $t\in\mathbb{R}$ and
$$\gamma_t\colon [0,1]\to\mathbb{R}^2,\ s\mapsto (as+b, cs+d),$$
where $a,b,c,d\in\mathbb{R}$ such that
$$\gamma_t(0) = (b,d) = (0,1) \quad \text{und} \quad \gamma_t(1) = (a+b, c+d) = (t,0),$$
and hence
$$\gamma_t(s) = (ts, -s+1)$$
for all $s\in[0,1]$. Furthermore, we have
$$\begin{aligned}|\gamma_t(s)|^2 &= |ts|^2 + |-s+1|^2 = s^2t^2 + (1-s)^2 = s^2t^2 + 1 - 2s + s^2\\ &= s(s(t^2+1)-2)+1 = 1\end{aligned}$$
if and only if
$$s = 0 \quad \text{or} \quad s = \frac{2}{t^2+1}.$$
Thus we obtain
$$x\colon\mathbb{R}\to\mathbb{R},\ t\mapsto\frac{2t}{t^2+1}$$
and
$$y\colon\mathbb{R}\to\mathbb{R},\ t\mapsto 1-\frac{2}{t^2+1} = \frac{t^2-1}{t^2+1},$$
hence
$$\alpha\colon\mathbb{R}\to\mathbb{R}^2,\ t\mapsto (x(t),y(t)) = \left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right).$$
Since
$$\alpha'(t) = \frac{2}{(t^2+1)^2}(1-t^2, 2t) \neq 0$$

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for all $t \in \mathbb{R}$, α is a regular parametrization.

It is left to show that $\mathrm{Im}(\alpha) = K \setminus \{(0,1)\}$. To this end, let $(\tilde{x}, \tilde{y}) \in K \setminus \{(0,1)\}$. We are looking for a $t \in \mathbb{R}$ such that

$$(x(t), y(t)) = (\tilde{x}, \tilde{y}).$$

Assume that there exists $t \in \mathbb{R}$ such that

$$(x(t), y(t)) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = (\tilde{x}, \tilde{y}).$$

For $\tilde{x} = 0$, i.e. $(\tilde{x}, \tilde{y}) = (0, -1)$, $t = 0$ is the unique solution. For $\tilde{x} \neq 0$, the above system of equations has the unique solution

$$t = \frac{1+\tilde{y}}{\tilde{x}},$$

hence $\mathrm{Im}(\alpha) = K \setminus \{(0,1)\}$.

(ii) For $t \in \mathbb{R}$, since

$$\alpha'(t) = \frac{2}{(t^2+1)^2}(1-t^2, 2t),$$

we have

$$\begin{aligned}
|\alpha'(t)| &= \frac{2}{(t^2+1)^2}\sqrt{(t^2-1)^2+4t^2} \\
&= \frac{2}{(t^2+1)^2}\sqrt{t^4-2t^2+2+4t^2} \\
&= \frac{2}{(t^2+1)^2}\sqrt{t^4+2t^2+1} \\
&= \frac{2}{t^2+1}.
\end{aligned}$$

It follows that

$$\int_{-1}^{1} |\alpha'(t)| \, \mathrm{d}t = 2\int_{-1}^{1} \frac{1}{t^2+1} \, \mathrm{d}t = 2[\arctan(t)]_{-1}^{1} = \pi.$$

Exercise 2.

Justify that the following curves in $\mathbb{R}^3$ have finite arc lengths and calculate them:

(i) $\beta\colon [0,1] \to \mathbb{R}^3,\ t \mapsto (6t, 3t^2, t^3)$,

(ii) $\gamma\colon [0, \sqrt{2}] \to \mathbb{R}^3,\ t \mapsto (t, t\sin(t), t\cos(t))$.

(*Hint: You can use the following identity without proving it:* $\int_0^s \sqrt{1+t^2}\, \mathrm{d}t = \frac{1}{2}(\sqrt{1+s^2} \cdot s + \operatorname{arsinh}(s))$ *with* $s > 0$.)

Solution 2.

(i) We have

$$\beta'(t) = (6, 6t, 3t^2) = 3(2, 2t, t^2)$$

for all $t \in [0,1]$ and hence

$$|\beta'(t)| = 3\sqrt{4+4t^2+t^4} = 3(2+t^2)$$

for all $t \in [0,1]$. Therefore, we obtain that

$$\int_0^1 |\beta'(t)| \, \mathrm{d}t = 3\int_0^1 2+t^2 \, \mathrm{d}t = 3\left[2t + \frac{1}{3}t^3\right]_0^1 = 7.$$

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(ii) We have

$$\gamma'(t) = (1, \sin(t) + t\cos(t), \cos(t) - t\sin(t))$$

for all $t \in [0, \sqrt{2}]$ and hence

$$\begin{aligned}
|\gamma'(t)|^2 &= 1 + (\sin(t) + t\cos(t))^2 + (\cos(t) - t\sin(t))^2 \\
&= 1 + \sin(t)^2 + 2\sin(t)\cos(t)t + (t\cos(t))^2 + \cos(t)^2 \\
&\quad - 2t\cos(t)\sin(t) + (t\sin(t))^2 \\
&= 2 + t^2 \\
&= 2\left(1 + \left(\frac{t}{\sqrt{2}}\right)^2\right)
\end{aligned}$$

for all $t \in [0, \sqrt{2}]$. Using the hint, we obtain

$$\begin{aligned}
\int_0^{\sqrt{2}} |\gamma'(t)| \,\mathrm{d}t &= \sqrt{2}\int_0^{\sqrt{2}} \sqrt{1 + \left(\frac{t}{\sqrt{2}}\right)^2} \,\mathrm{d}t \\
&= 2\int_0^1 \sqrt{1 + t^2} \,\mathrm{d}t \\
&= \sqrt{2} + \operatorname{arsinh}(1).
\end{aligned}$$

Exercise 3.

Reparameterize the following curves by arc length: (*Hint: Remark 2 on p. 23 in [Car16] can be useful.*)

(i) $\delta\colon (1, \infty) \to \mathbb{R}^3,\ t \mapsto e^{-t}(\cos(t), \sin(t), 1)$,

(ii) $\varepsilon\colon (0, \infty) \to \mathbb{R}^3,\ t \mapsto (e^t, e^{-t}, \sqrt{2}t)$.

Solution 3.

(i) We have

$$\begin{aligned}
\delta'(t) &= -e^{-t}(\cos(t), \sin(t), 1) + e^{-t}(-\sin(t), \cos(t), 0) \\
&= e^{-t}(-\cos(t) - \sin(t), \cos(t) - \sin(t), -1)
\end{aligned}$$

for all $t \in (1, \infty)$ and hence

$$|\delta'(t)| = e^{-t}\sqrt{(\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2 + 1} = \sqrt{3}e^{-t}$$

for all $t \in (1, \infty)$. Therefore we obtain

$$\int_1^t |\delta'(\tau)| \,\mathrm{d}\tau = \sqrt{3}[-e^{-\tau}]_1^t = \sqrt{3}(-e^{-t} + e^{-1})$$

for all $t \in (1, \infty)$ and finally

$$s\colon (1, \infty) \to \mathbb{R},\ t \mapsto \int_1^t |\delta'(\tau)| \,\mathrm{d}\tau = \sqrt{3}(-e^{-t} + e^{-1}).$$

The inverse function of s restricted to its image is given by

$$\varphi = s^{-1}\colon \left(0, \frac{\sqrt{3}}{e}\right) \to (1, \infty),\ t \mapsto -\ln\left(-\left(\frac{t}{\sqrt{3}} - e^{-1}\right)\right).$$

Therefore,

$$\overline{\delta}\colon \left(0, \frac{\sqrt{3}}{e}\right) \to \mathbb{R}^3,\ t \mapsto \delta(\varphi(t))$$

is the desired reparametrization by arc length.

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(ii) We have

$$\varepsilon'(t) = (e^t, -e^{-t}, \sqrt{2})$$

for all $t \in (0, \infty)$ and hence

$$|\varepsilon'(t)|^2 = e^{2t} + e^{-2t} + 2 = e^{-2t}(e^{4t} + 2e^{2t} + 1) = e^{-2t}(e^{2t} + 1)^2 = (e^t + e^{-t})^2$$

Therefore we obtain

$$\int_0^t |\varepsilon'(\tau)| \, \mathrm{d}\tau = \int_0^t e^\tau + e^{-\tau} \, \mathrm{d}\tau = [e^\tau - e^{-\tau}]_0^t = 2\sinh(t)$$

for all $t \in (0, \infty)$ and finally

$$s\colon (0, \infty) \to (0, \infty), \ t \mapsto \int_0^t |\varepsilon'(\tau)| \, \mathrm{d}\tau = 2\sinh(t).$$

The inverse function of s restricted to its image is given by

$$\varphi = s^{-1}\colon (0, \infty) \to (0, \infty), \ t \mapsto \operatorname{arsinh}\left(\frac{t}{2}\right).$$

Therefore,

$$\overline{\varepsilon}\colon (0, \infty) \to \mathbb{R}^3, \ t \mapsto \varepsilon(\varphi(t))$$

is the desired reparametrization by arc length.

Exercise 4.
Let $\alpha\colon I \to \mathbb{R}^3$ ($I \subset \mathbb{R}$ an interval) be a regular curve, $[a, b] \subset I$ and $A = \alpha(a)$, $B = \alpha(b)$ with $A \neq B$. Show:

(i) For each unit vector $e \in \mathbb{R}^3$, we have

$$(B - A) \cdot e \leq L_\alpha,$$

where L_α is the arc length between A and B with respect to α.

(ii) The shortest arc length of any curve connecting A and B is the straight line connecting them.

Solution 4.

(i) Let $e \in \mathbb{R}^3$ be a unit vector. By the mean value theorem, we have

$$\alpha(b) - \alpha(a) = \left(\int_0^1 \alpha'(a + t(b - a)) \, \mathrm{d}t\right)(b - a) = \int_a^b \alpha'(t) \, \mathrm{d}t.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(B - A) \cdot e = (\alpha(b) - \alpha(a)) \cdot e &= \left(\int_a^b \alpha'(t) \, \mathrm{d}t\right) \cdot e \leq \left|\int_a^b \alpha'(t) \, \mathrm{d}t \cdot e\right| \\ &\leq \left|\int_a^b \alpha'(t) \, \mathrm{d}t\right| |e| \leq \int_a^b |\alpha'(t)| \, \mathrm{d}t = L_\alpha.\end{aligned}$$

(ii) Let

$$\gamma\colon [0, 1] \to \mathbb{R}^2, \ t \mapsto \alpha(a) + t(\alpha(b) - \alpha(a))$$

and

$$e = \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}.$$

Then we have

$$L_\gamma = |\alpha(b) - \alpha(a)| = (\alpha(b) - \alpha(a)) \cdot e = (B - A) \cdot e \leq L_\alpha.$$

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References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.