UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 2 **Submission:** /

Resources: §1 – §2; Sections 1-1 – 1-6 in [Car16]

Exercise 1.
Let $\alpha, \beta \colon I \to \mathbb{R}^3$ be differentiable functions on an interval $I \subset \mathbb{R}$. Show:

(i) The function $\alpha \times \beta \colon I \to \mathbb{R}^3$ is differentiable with

$$(\alpha \times \beta)' = \alpha' \times \beta + \alpha \times \beta'.$$

(ii) For constants $a, b, c \in \mathbb{R}$, if the relations

$$\alpha' = a\alpha + b\beta \quad \text{and} \quad \beta' = c\alpha - a\beta$$

hold, then $\alpha \times \beta$ is constant.

(iii) For $u, v, w \in \mathbb{R}^3$, the identity

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$$

holds.

Exercise 2.
Let $a, b, c \in \mathbb{R}$ with $a^2 + b^2 = c^2$ and $a \neq 0$. Consider the following parameterized curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^3,\ s \mapsto \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right).$$

(i) Is γ parameterized by arc length?

(ii) Calculate the curvature and torsion of γ.

(iii) Show that the angle under which the line containing $n_\gamma(s)$ and passing through $\gamma(s)$ meets the z axis is independent of $s \in \mathbb{R}$. Calculate this angle.

(iv) Plot the curve of γ.

Exercise 3.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha \colon I \to \mathbb{R}^3$ be a (not necessarily parameterized by arc length) regular curve with nowhere vanishing curvature. Show that the Frenet trihedron $(t_\alpha, n_\alpha, b_\alpha)$ is given by

$$t_\alpha = \frac{\alpha'}{|\alpha'|},\ n_\alpha = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \times \frac{\alpha'}{|\alpha'|},\ b_\alpha = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}.$$

(*Hint: Reparameterize the curve by arc length and use Exercise 1. Without a proof, you can use Exercise 12 a–c in Section 1-5 in [Car16]*)

(please turn the page)

Exercise 4.

(i) Show that the signed curvature of a regular plane curve $\alpha\colon I \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ ($I \subset \mathbb{R}$ an interval) is given by

$$\kappa_\alpha\colon I \to \mathbb{R},\ t \mapsto \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}.$$

(ii) Show that a change of orientation changes the sign of the signed curvature of a regular plane curve.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.