UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 2, Solution **Submission:** /

Resources: §1 – §2; Sections 1-1 – 1-6 in [Car16]

Exercise 1.
Let $\alpha, \beta \colon I \to \mathbb{R}^3$ be differentiable functions on an interval $I \subset \mathbb{R}$. Show:

(i) The function $\alpha \times \beta \colon I \to \mathbb{R}^3$ is differentiable with

$$(\alpha \times \beta)' = \alpha' \times \beta + \alpha \times \beta'.$$

(ii) For constants $a, b, c \in \mathbb{R}$, if the relations

$$\alpha' = a\alpha + b\beta \quad \text{and} \quad \beta' = c\alpha - a\beta$$

hold, then $\alpha \times \beta$ is constant.

(iii) For $u, v, w \in \mathbb{R}^3$, the identity

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$$

holds.

Solution 1.
Simple calculations.

Exercise 2.
Let $a, b, c \in \mathbb{R}$ with $a^2 + b^2 = c^2$ and $a \neq 0$. Consider the following parameterized curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^3,\ s \mapsto \left(a \cos\left(\frac{s}{c}\right), a \sin\left(\frac{s}{c}\right), b\frac{s}{c}\right).$$

(i) Is γ parameterized by arc length?

(ii) Calculate the curvature and torsion of γ.

(iii) Show that the angle under which the line containing $n_\gamma(s)$ and passing through $\gamma(s)$ meets the z axis is independent of $s \in \mathbb{R}$. Calculate this angle.

(iv) Plot the curve of γ.

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Solution 2.

(i) Let $s \in \mathbb{R}$. We have

$$\gamma'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right) = \frac{1}{c}\left(-a\sin\left(\frac{s}{c}\right), a\cos\left(\frac{s}{c}\right), b\right),$$

and hence

$$|\gamma'(s)|^2 = \left(\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)^2 + \left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2+b^2}{c^2} = 1.$$

Therefore, γ is parameterized by arc length.

(ii) Let $s \in \mathbb{R}$. We have

$$\gamma''(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right) = -\frac{a}{c^2}\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right),$$

and hence

$$\kappa_\gamma(s) = |\gamma''(s)| = \frac{|a|}{c^2}.$$

Furthermore, we have

$$n_\gamma(s) = \frac{\gamma''(s)}{\kappa_\gamma(s)} = -\frac{a}{|a|}\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right) = -\operatorname{sgn}(a)\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right)$$

and

$$n'_\gamma(s) = -\frac{\operatorname{sgn}(a)}{c}\left(-\sin\left(\frac{s}{c}\right), \cos\left(\frac{s}{c}\right), 0\right)$$

as well as (see the calculation on p. 19 in [Car16])

$$\begin{aligned}
b'_\gamma(s) &= \gamma'(s) \times n'_\gamma(s) \\
&= -\frac{\operatorname{sgn}(a)}{c^2}\left(-a\sin\left(\frac{s}{c}\right), a\cos\left(\frac{s}{c}\right), b\right) \times \left(-\sin\left(\frac{s}{c}\right), \cos\left(\frac{s}{c}\right), 0\right) \\
&= -\frac{\operatorname{sgn}(a)}{c^2}\left(-b\cos\left(\frac{s}{c}\right), -b\sin\left(\frac{s}{c}\right), 0\right) \\
&= -\frac{b}{c^2} n_\gamma(s),
\end{aligned}$$

thus

$$\tau_\gamma(s) = -\frac{b}{c^2}.$$

(iii) Let $s \in \mathbb{R}$ and define

$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \gamma(s) + t n_\gamma(s) = \left(a\cos\left(\frac{s}{c}\right)\left(1 - t\frac{1}{|a|}\right), a\sin\left(\frac{s}{c}\right)\left(1 - t\frac{1}{|a|}\right), b\frac{s}{c}\right).$$

The line α meets the z axis at $t = |a|$. Since

$$n_\gamma(s) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,$$

the desired angle equals $\pi/2$.

(iv) Cf. Figure 1-1 on p. 3 in [Car16] (Helix).

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Exercise 3.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^3$ be a (not necessarily parameterized by arc length) regular curve with nowhere vanishing curvature. Show that the Frenet trihedron $(t_\alpha, n_\alpha, b_\alpha)$ is given by

$$t_\alpha = \frac{\alpha'}{|\alpha'|},\ n_\alpha = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \times \frac{\alpha'}{|\alpha'|},\ b_\alpha = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}.$$

(*Hint: Reparameterize the curve by arc length and use Exercise 1. Without a proof, you can use Exercise 12 (a)–(c) in Section 1-5 in [Car16]*)

Solution 3.
Let s_α be the arc length of α and let $\varphi\colon J = \mathrm{Im}(s_\alpha) \to I$ be the inverse function of s_α (restricted to its image). Consider now the reparameterized regular curve

$$\tilde{\alpha}\colon J \to \mathbb{R}^3,\ \tau \mapsto (\alpha \circ \varphi)(\tau) = \alpha(\varphi(\tau)).$$

By definition, we have

$$t_\alpha = t_{\tilde{\alpha}} \circ \varphi^{-1},\ n_\alpha = n_{\tilde{\alpha}} \circ \varphi^{-1} \text{ and } b_\alpha = b_{\tilde{\alpha}} \circ \varphi^{-1}.$$

By Exercise 12 (a) in Section 1-5 in [Car16], we have

$$\varphi' = (|\alpha'|^{-1}) \circ \varphi$$

and

$$\varphi'' = (-|\alpha'|^{-4}(\alpha'' \cdot \alpha')) \circ \varphi.$$

Thus

$$t_{\tilde{\alpha}} = \tilde{\alpha}' = (\alpha' \circ \varphi)\varphi' = (\alpha' \circ \varphi)|(\alpha' \circ \varphi)|^{-1} = \left(\frac{\alpha'}{|\alpha'|}\right) \circ \varphi$$

and, using Exercise 1 (iii), we obtain

$$\begin{aligned}
\tilde{\alpha}'' &= (\alpha' \circ \varphi)'\varphi' + (\alpha' \circ \varphi)\varphi'' \\
&= (\alpha'' \circ \varphi)(\varphi')^2 + (\alpha' \circ \varphi)\varphi'' \\
&= \left(\frac{\alpha''}{|\alpha'|^2} - \frac{\alpha'\alpha'' \cdot \alpha'}{|\alpha'|^4}\right) \circ \varphi \\
&= \left(\frac{\alpha''|\alpha'|^2 - \alpha'\alpha'' \cdot \alpha'}{|\alpha'|^4}\right) \circ \varphi \\
&= \left(\frac{(\alpha' \times \alpha'') \times \alpha'}{|\alpha'|^4}\right) \circ \varphi.
\end{aligned}$$

By Exercise 12 (b) in Section 1-5 in [Car16], we conclude that

$$n_{\tilde{\alpha}} = \frac{\tilde{\alpha}''}{\kappa_{\tilde{\alpha}}} = \left(\frac{(\alpha' \times \alpha'') \times \alpha'}{|\alpha'|^4} \frac{|\alpha'|^3}{|\alpha' \times \alpha''|}\right) \circ \varphi = \left(\frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \times \frac{\alpha'}{|\alpha'|}\right) \circ \varphi.$$

Finally, we obtain with Exercise 1 (iii) that

$$\begin{aligned}
b_{\tilde{\alpha}} = t_{\tilde{\alpha}} \times n_{\tilde{\alpha}} &= \left(\frac{\alpha'}{|\alpha'|} \times (\frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \times \frac{\alpha'}{|\alpha'|})\right) \circ \varphi \\
&= \left(-\frac{1}{|\alpha' \times \alpha''||\alpha'|^2}(\alpha' \times (\alpha'' \times \alpha')) \times \alpha'\right) \circ \varphi \\
&= \left(-\frac{1}{|\alpha' \times \alpha''||\alpha'|^2}((\alpha' \cdot \alpha')(\alpha'' \times \alpha') - ((\alpha'' \times \alpha') \cdot \alpha')\alpha')\right) \circ \varphi \\
&= \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} \circ \varphi,
\end{aligned}$$

since $(\alpha'' \times \alpha') \cdot \alpha' = 0$.

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Exercise 4.

(i) Show that the signed curvature of a regular plane curve $\alpha\colon I \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ ($I \subset \mathbb{R}$ an interval) is given by

$$\kappa_\alpha\colon I \to \mathbb{R},\ t \mapsto \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}.$$

(ii) Show that a change of orientation changes the sign of the signed curvature of a regular plane curve.

Solution 4.

Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ be a regular plane curve.

(i) Let s_α be the arc length of α and let $\varphi\colon J = \mathrm{Im}(s_\alpha) \to I$ be the inverse function of s_α (restricted to its image). Consider now the reparameterized regular curve

$$\tilde{\alpha}\colon J \to \mathbb{R}^2,\ \tau \mapsto (\alpha \circ \varphi)(\tau) = \alpha(\varphi(\tau)).$$

With $t_{\tilde{\alpha}} = \tilde{\alpha}'$ and

$$\tilde{n}_{\tilde{\alpha}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t_{\tilde{\alpha}} = \left(\frac{1}{|\alpha'|} \begin{pmatrix} -y' \\ x' \end{pmatrix} \right) \circ \varphi,$$

$(t_{\tilde{\alpha}}, \tilde{n}_{\tilde{\alpha}})$ is pointwise a positive oriented orthonormal basis. We have (cf. the solution of Exercise 3)

$$\begin{aligned}
t'_{\tilde{\alpha}} = \tilde{\alpha}'' &= \left(\frac{\alpha''|\alpha'|^2 - \alpha'\alpha'' \cdot \alpha'}{|\alpha'|^4} \right) \circ \varphi \\
&= \left(\frac{1}{|\alpha'|^4} \left((x'^2 + y'^2) \begin{pmatrix} x'' \\ y'' \end{pmatrix} - (x''x' + y''y) \begin{pmatrix} x' \\ y' \end{pmatrix} \right) \right) \circ \varphi \\
&= \left(\frac{1}{|\alpha'|^4} \begin{pmatrix} x''y'^2 - y''y'x' \\ y''x'^2 - y'x''x' \end{pmatrix} \right) \circ \varphi \\
&= \left(\frac{x'y'' - x''y'}{|\alpha'|^3} \left(\frac{1}{|\alpha'|} \begin{pmatrix} -y' \\ x' \end{pmatrix} \right) \right) \circ \varphi \\
&= \left(\left(\frac{x'y'' - x''y'}{|\alpha'|^3} \right) \circ \varphi \right) \tilde{n}_{\tilde{\alpha}},
\end{aligned}$$

and hence

$$\kappa_\alpha = \kappa_{\tilde{\alpha}} \circ \varphi^{-1} = \frac{x'y'' - x''y'}{|\alpha'|^3}.$$

(ii) Let Ψ be the change of orientation of α ($\Psi' = -1$) and let $\alpha_\Psi = (x_\Psi, y_\Psi)$ be the reoriented curve with respect to α. By part (i), we have

$$\kappa_{\alpha_\Psi} = \frac{x'_\Psi y''_\Psi - x''_\Psi y'_\Psi}{|\alpha'_\Psi|^3} = \left(-\frac{x'y'' - x''y'}{|\alpha'|^3} \right) \circ \Psi = (-\kappa_\alpha) \circ \Psi.$$

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.