UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 3 **Submission:** /

Resources: Lessons 1 – 7; §1 – §2 in [Fuc08]; Sections 1-1 – 1-6 in [Car16]

Exercise 1.
By the *Picard-Lindelöf theorem* we have the following: Is $J \subset \mathbb{R}$ a *compact* interval and $F\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous with respect to the second component, i.e. there exists a constant $L > 0$ such that, for all $t \in J, y_1, y_2 \in \mathbb{R}^n$, we have

$$|F(t, y_1) - F(t, y_2)| \leq L|y_1 - y_2|,$$

then there exists a unique solution $y\colon J \to \mathbb{R}^n$ to the following system of ordinary differential equations

$$\dot{y} = F(\cdot, y).$$

Show: The above statement still holds under the assumptions that $I \subset \mathbb{R}$ is an arbitrary interval and $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and *linear* in the second component.

(*Hint: Use an exhaustion by compact sets of the interval I.*)

Exercise 2.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha \in C^2(I, \mathbb{R}^3)$ be a regular parametrized curve. Show:

(i) If all normals of the curve intersect in one point, then the trace is part of a circle.

(ii) If all tangents of the curve intersect in one poirnt, then the trace is part of a straight line. Does this still hold without the regularity assumption on α?

Exercise 3.
For $r > 0$, conside the function

$$\gamma\colon (-\pi, \pi) \to \mathbb{R}^3,\ t \mapsto r\left(1 + \cos(t), \sin(t), 2\sin\left(\frac{t}{2}\right)\right).$$

Show:

(i) The curve γ lies in the intersection of the cylinder $\{(x, y, z) \in \mathbb{R}^3\ ;\ (x - r)^2 + y^2 = r^2\}$ and the sphere around the origin with radius $2r$.

(ii) Calculate the Frenet trihedron of γ.

(iii) Calculate the curvature and the torsion of γ.

(*Hint: Trigonometric identites can be useful.*)

(please turn the page)

Exercise 4.

(i) Let $I \subset \mathbb{R}$ be an interbval, let $s_0 \in I$ and let $\kappa\colon I \to \mathbb{R}$ be a differentiable function. Show that

$$\alpha\colon I \to \mathbb{R}^2,\ s \mapsto \left(\int_{s_0}^{s} \cos(\theta(t))\,\mathrm{d}t + a, \int_{s_0}^{s} \sin(\theta(t))\,\mathrm{d}t + b\right)$$

with

$$\theta\colon I \to \mathbb{R},\ s \mapsto \int_{s_0}^{s} \kappa(t)\,\mathrm{d}t + \varphi$$

is a regular curve which is paramterized by arc length and such that κ is the oriented curvature. Furthermore, show that this curve is unique up to a translation of the vector $(a, b) \in \mathbb{R}^2$ and a rotation of the angle φ.

(ii) A so-called *clothoid* is a planar curve which is determined by the fact that the curvature in each point is proportional to its arc length up to this point. Determine the regular paramterization $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ of a clothoid with $\alpha(0) = (0, 0)$ and $\alpha'(0) = (1, 0)$.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.