UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 3, Solution **Submission:** /

Resources: Lessons 1 – 7; §1 – §2 in [Fuc08]; Sections 1-1 – 1-6 in [Car16]

Exercise 1.
By the *Picard-Lindelöf theorem* we have the following: Is $J \subset \mathbb{R}$ a *compact* interval and $F\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous with respect to the second component, i.e. there exists a constant $L > 0$ such that, for all $t \in J, y_1, y_2 \in \mathbb{R}^n$, we have

$$|F(t, y_1) - F(t, y_2)| \leq L|y_1 - y_2|,$$

then there exists a unique solution $y\colon J \to \mathbb{R}^n$ to the following system of ordinary differential equations

$$\dot{y} = F(\cdot, y).$$

Show: The above statement still holds under the assumptions that $I \subset \mathbb{R}$ is an arbitrary interval and $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and *linear* in the second component.

(*Hint: Use an exhaustion by compact sets of the interval I.*)

Solution 1.
Let $I \subset \mathbb{R}$ be an arbitrary interval and let $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and linear in the second component. Then we have

$$F(t, y) = F(t, 1)y$$

for all $(t, y) \in I \times \mathbb{R}^n$. Since $F(\cdot, 1)$ is continuous, there exists $\|F(\cdot, 1)\|_K$ for each compact interval $K \subset I$, hence F is Lipschitz continuous on $K \times \mathbb{R}^n$. Choose an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact intervals with $\cup_{n\in\mathbb{N}} K_n = I$. For all n, there exists a unique solution y_n to the initial value problem on K_n. By

$$y(x) = y_n(x)$$

for $x \in K_n$, we can define a function $y\colon I \to \mathbb{R}^n$ which is the unique global solution to the initial value problem.

Exercise 2.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha \in C^2(I, \mathbb{R}^3)$ be a regular parametrized curve. Show:

(i) If all normals of the curve intersect in one point, then the trace is part of a circle.

(ii) If all tangents of the curve intersect in one poirnt, then the trace is part of a straight line. Does this still hold without the regularity assumption on α?

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Solution 2.
Without loss of generality, let α be parameterized by arc length.

(i) By assumption, there exists a differentiable function

$$\lambda\colon I \to \mathbb{R}$$

such that $\alpha(s) + \lambda(s)n_\alpha(s)$ is constant for all $s \in I$. With the Frenet formulas we obtain

$$0 = \alpha' + \lambda' n_\alpha + \lambda n'_\alpha = (1 - \lambda\kappa_\alpha)t_\alpha + \lambda' n_\alpha + (-1\lambda\tau_\alpha)b_\alpha.$$

Since t, n, b is linear independent, we conclude that

$$(1 - \lambda\kappa_\alpha) = 0, \quad \lambda' = 0 \quad \text{und} \quad \lambda\tau_\alpha = 0.$$

Hence λ is constant and therefore $\kappa_\alpha = 1/\lambda$ as well as $\tau_\alpha = 0$. Thus we see that α is planar with constant curvature, i.e. the trace of α is part of a circle.

(ii) By assumption, there exists a differentiable function

$$\lambda\colon I \to \mathbb{R},$$

such that $\alpha(s) + \lambda(s)t_\alpha(s)$ is constant for all $s \in I$. With the Frenet formulas we obtain

$$0 = \alpha' + \lambda' t_\alpha + \lambda t'_\alpha = (1 + \lambda')t_\alpha + (\lambda\kappa_\alpha)n_\alpha.$$

Since t and n are linear independent, it follows that

$$(1 + \lambda') = 0 \quad \text{und} \quad \lambda\kappa_\alpha = 0.$$

Hence $\lambda(s) = -s + a$ for all $s \in I$ with a constant $a \in \mathbb{R}$ and therefore $\kappa_\alpha(s) = 0$ for all $s \neq a$. Since κ_α is continuous, we conclude that $\kappa_\alpha = 0$ and thus $\alpha'' = 0$. Therefore the trace of α is part of a straight line.

Consider

$$\alpha\colon (-1, 1) \to \mathbb{R}^3,\ t \mapsto \begin{cases} (t, -t, 0), & \text{if } t < 0, \\ (t, t, 0), & \text{if } t \geq 0. \end{cases}$$

Then all tangents intersect in the origin but the trace of the curve is not a part of a straight line.

Exercise 3.
For $r > 0$, conside the function

$$\gamma\colon (-\pi, \pi) \to \mathbb{R}^3,\ t \mapsto r\left(1 + \cos(t), \sin(t), 2\sin\left(\frac{t}{2}\right)\right).$$

Show:

(i) The curve γ lies in the intersection of the cylinder $\{(x, y, z) \in \mathbb{R}^3\ ;\ (x - r)^2 + y^2 = r^2\}$ and the sphere around the origin with radius $2r$.

(ii) Calculate the Frenet trihedron of γ.

(iii) Calculate the curvature and the torsion of γ.

(*Hint: Trigonometric identites can be useful.*)

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Solution 3.

(i) This follows immediately from the following two identites:

$$\sin(t)^2 + \cos(t)^2 = 1$$

and

$$2\sin\left(\frac{t}{2}\right)^2 = 1 - \cos(t)$$

for all $t \in (-\pi, \pi)$.

(ii) Let $t \in (-\pi, \pi)$. Then we have

$$\gamma'(t) = r \begin{pmatrix} -\sin(t) \\ \cos(t) \\ \cos(\frac{t}{2}) \end{pmatrix}$$

and

$$\gamma''(t) = -r \begin{pmatrix} \cos(t) \\ \sin(t) \\ \frac{1}{2}\sin(\frac{t}{2}) \end{pmatrix}.$$

Hence

$$\begin{aligned} \gamma'(t) \times \gamma''(t) &= -r^2 \begin{pmatrix} (\cos(t)\frac{1}{2}\sin(\frac{t}{2})) - (\cos(\frac{t}{2})\sin(t)) \\ (\cos(\frac{t}{2})\cos(t)) - (-\sin(t)\frac{1}{2}\sin(\frac{t}{2})) \\ (-\sin(t)\sin(t)) - (\cos(t)\cos(t)) \end{pmatrix} \\ &= -r^2 \begin{pmatrix} \cos(t)\frac{1}{2}\sin(\frac{t}{2}) - \cos(\frac{t}{2})\sin(t) \\ \cos(\frac{t}{2})\cos(t) + \sin(t)\frac{1}{2}\sin(\frac{t}{2})) \\ -1 \end{pmatrix} \\ &= r^2 \begin{pmatrix} \frac{1}{2}\sin(\frac{t}{2})(\cos(t)+2) \\ -\cos(\frac{t}{2})^3 \\ 1 \end{pmatrix}. \end{aligned}$$

With

$$|\gamma'(t)|^2 = r^2\left(1 + \cos\left(\frac{t}{2}\right)^2\right) = r^2\frac{1}{2}(\cos(t)+3)$$

it follows that

$$t_\gamma(t) = \sqrt{\frac{2}{\cos(t)+3}} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ \cos(\frac{t}{2}) \end{pmatrix}$$

and with

$$|\gamma'(t) \times \gamma''(t)|^2 = \frac{r^4}{8}(3\cos(t)+13)$$

it follows that

$$b_\gamma(t) = \sqrt{\frac{8}{3\cos(t)+13}} \begin{pmatrix} \frac{1}{2}\sin(\frac{t}{2})(\cos(t)+2) \\ -\cos(\frac{t}{2})^3 \\ 1 \end{pmatrix}$$

Hence we obtain

$$\begin{aligned} n_\gamma(t) &= b_\gamma(t) \times t_\gamma(t) \\ &= \frac{4}{\sqrt{(3\cos(t)+13)(\cos(t)+3)}} \begin{pmatrix} -\cos\left(\frac{t}{2}\right)^4 - \cos(t) \\ -\frac{1}{4}\sin(t)(\cos(t)+6) \\ \frac{1}{4}\sin(t)(-4\cos\left(\frac{t}{2}\right)^3 + \cos(t)^2 + 2\cos(t)) \end{pmatrix}. \end{aligned}$$

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(iii) Let $t \in (-\pi, \pi)$. Then we have

$$\kappa_\gamma(t) = \frac{1}{|\gamma'(t)|^3}|\gamma'(t) \times \gamma''(t)| = \frac{1}{r}\sqrt{\frac{3\cos(t)+13}{(\cos(t)+3)^3}}.$$

Furthermore, we obtain

$$\gamma'''(t) = -r\begin{pmatrix} -\sin(t) \\ \cos(t) \\ \frac{1}{4}\cos(\frac{t}{2}) \end{pmatrix},$$

thus

$$\tau_\gamma(t) = -\frac{\gamma'(t) \times \gamma''(t)}{|\gamma'(t) \times \gamma''(t)|^2} \cdot \gamma'''(t) = -\frac{6}{r}\frac{\cos\left(\frac{t}{2}\right)}{3\cos(t)+13}.$$

Exercise 4.

(i) Let $I \subset \mathbb{R}$ be an interbval, let $s_0 \in I$ and let $\kappa\colon I \to \mathbb{R}$ be a differentiable function. Show that

$$\alpha\colon I \to \mathbb{R}^2,\ s \mapsto \left(\int_{s_0}^{s} \cos(\theta(t))\,\mathrm{d}t + a, \int_{s_0}^{s} \sin(\theta(t))\,\mathrm{d}t + b\right)$$

with

$$\theta\colon I \to \mathbb{R},\ s \mapsto \int_{s_0}^{s} \kappa(t)\,\mathrm{d}t + \varphi$$

is a regular curve which is paramterized by arc length and such that κ is the oriented curvature. Furthermore, show that this curve is unique up to a translation of the vector $(a,b) \in \mathbb{R}^2$ and a rotation of the angle φ.

(ii) A so-called *clothoid* is a planar curve which is determined by the fact that the curvature in each point is proportional to its arc length up to this point. Determine the regular paramterization $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ of a clothoid with $\alpha(0) = (0,0)$ and $\alpha'(0) = (1,0)$.

Solution 4.

(i) Let $s \in I$. We have

$$\alpha'(s) = (\cos(\theta(s)), \sin(\theta(s))) = (\cos(\theta(s)), \sin(\theta(s)))\,,$$

hence

$$|\alpha'(s)|^2 = \cos(\theta(s))^2 + \sin(\theta(s))^2 = 1.$$

Furthermore, we obtain

$$\alpha''(s) = (-\sin(\theta(s))\theta'(s), \cos(\theta(s))\theta'(s)) = \kappa(s)(-\sin(\theta(s)), \cos(\theta(s)))$$

and with Exercise 4, Sheet 2

$$\tilde{\kappa}(s) = \kappa(s)(\cos(\theta(s))\cos(\theta(s)) - (-\sin(\theta(s)))\sin(\theta(s))) = \kappa(s).$$

The uniqueness is clear by integration and the independence of $|\alpha'|$ and $\tilde{\kappa}$ from θ

(ii) Choose $s_0 = 0$. By assumptoon, there exists $c \in \mathbb{R}$ such that

$$\kappa(s) = c \cdot s$$

for all $s \in I$. Thus

$$\theta(s) = \frac{1}{2}cs^2 + \varphi$$

for all $s \in I$. The condition $\alpha(0) = (0,0)$ implies $(a,b) = (0,0)$. Since

$$\alpha'(0) = (\cos(\varphi), \sin(\varphi)),$$

the condtion$\alpha'(0) = (1,0)$ implies that $\varphi = 0$. Finally, we obtain that

$$\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ s \mapsto \left(\int_0^s \cos\left(\frac{1}{2}ct^2\right)\,\mathrm{d}t, \int_0^s \sin\left(\frac{1}{2}ct^2\right)\,\mathrm{d}t\right).$$

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References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.