UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 4 **Submission:** /

Materialien: up to Lesson 8; up to p. 39 in [Fuc08]; up to Section 1-7 B in [Car16]

Exercise 1.

(i) Let $\gamma\colon [a,b] \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ be a regular parameterized (not necessarily by arc length) planar curve. Show: The rotation index I_γ of γ satisfies

$$I_\gamma = \frac{1}{2\pi}\int_a^b \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2}\,\mathrm{d}t.$$

(ii) Plot the trace of the following planar curves and calculate the rotation indeces:

(a) $\alpha_n\colon [0,2\pi] \to \mathbb{R}^2,\ t \mapsto (\cos(nt), \sin(nt))\ (n \in \mathbb{N})$,
(b) $\beta_{a,b}\colon [0,4\pi] \to \mathbb{R}^2,\ t \mapsto (a\cos(t), b\sin(t))\ (a,b > 0)$,
(c) $\gamma\colon [0,2\pi] \to \mathbb{R}^2,\ t \mapsto (\cos(t) - \cos(2t), \sin(t) - \sin(2t))$.

Exercise 2.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^3$ be a regular paramterized curve with nowhere vanishing curvature κ and nowhere vanishing torsion τ. Show the equivalence of the following statements:

(i) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ such that $t \cdot v$ is constant.

(ii) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with $n \cdot v \equiv 0$.

(iii) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ such that $b \cdot v$ is constant.

(iv) The ratio of the torsion τ and the curvature κ is constant.

A curve satisfying one of these equivalent conditions is called a *generalized helix*.

Exercise 3.
Consider the function

$$c\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{cases} (t, e^{-1/t^2}, 0), & \text{falls } t < 0, \\ (0,0,0), & \text{falls } t = 0, \\ (t, 0, e^{-1/t^2}), & \text{falls } t > 0. \end{cases}$$

(i) Show that c is a regular, two times continuously differentiable ($c \in C^2(\mathbb{R}, \mathbb{R}^3)$) curve.

(ii) Show that the curvature κ of c only vanishes on $\left\{0, \pm\sqrt{\frac{2}{3}}\right\}$. What does $\kappa(0) = 0$ means graphically?

(please turn the page)

(iii) Show that the limit of the osculating plane of c at $t \downarrow 0$ is the plane $\{(x, y, z) \; ; \; y = 0\}$, whereas at $t \uparrow 0$ the plane $\{(x, y, z) \; ; \; z = 0\}$ is approximated. What does that mean for the torsion?

Exercise 4.
Let $L > 0$ and let $\alpha\colon [0, L] \to \mathbb{R}^2$ be a planar, paramterized by arc length, simple closed curve. The curvature κ satisfies $0 < \kappa(s) \leq c$ for all $s \in [0, L]$ with a constant c. Show: The length L of the curve satisfies

$$L \geq \frac{2\pi}{c}.$$

What does that mean graphically?

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.