UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 4, Solution **Submission: /**

Materialien: up to Lesson 8; up to p. 39 in [Fuc08]; up to Section 1-7 B in [Car16]

Exercise 1.

(i) Let $\gamma\colon [a,b] \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ be a regular parameterized (not necessarily by arc length) planar curve. Show: The rotation index I_γ of γ satisfies

$$I_\gamma = \frac{1}{2\pi}\int_a^b \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2}\ \mathrm{d}t.$$

(ii) Plot the trace of the following planar curves and calculate the rotation indeces:

(a) $\alpha_n\colon [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto (\cos(nt), \sin(nt))\ (n \in \mathbb{N})$,

(b) $\beta_{a,b}\colon [0, 4\pi] \to \mathbb{R}^2,\ t \mapsto (a\cos(t), b\sin(t))\ (a, b > 0)$,

(c) $\gamma\colon [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto (\cos(t) - \cos(2t), \sin(t) - \sin(2t))$.

Solution 1.

(i) Let $\varphi\colon [0, L] \to [a, b]$ be the reparameterisation of γ by arc length and consider $\tilde{\gamma} = \gamma \circ \varphi$. By Exercise 4 (ii), Sheet 2 and integration by substitution, we have

$$\begin{aligned}
I_{\tilde{\gamma}} &= \frac{1}{2\pi}\int_0^L \kappa_{\tilde{\gamma}}(s)\ \mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^L \left(\left(\frac{x'y'' - x''y'}{|\gamma'|^3}\right) \circ \varphi\right)(s)\ \mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^L \left(\left(\frac{x'y'' - x''y'}{|\gamma'|^2}\right) \circ \varphi\right)(s) \left(\frac{1}{|\gamma'|} \circ \varphi\right)(s)\ \mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^L \left(\left(\frac{x'y'' - x''y'}{|\gamma'|^2}\right) \circ \varphi\right)(s)\varphi'(s)\ \mathrm{d}s \\
&= \frac{1}{2\pi}\int_a^b \left(\frac{x'y'' - x''y'}{|\gamma'|^2}\right)(t)\ \mathrm{d}t.
\end{aligned}$$

(ii) (a) Let $n \in \mathbb{N}$. Then we have

$$I_{\alpha_n} = \frac{1}{2\pi}\int_0^{2\pi} \frac{n^3\sin(nt)^2 + n^3\cos(nt)^2}{n^2}\ \mathrm{d}t = \frac{n}{2\pi}\int_0^{2\pi} 1\ \mathrm{d}t = n.$$

Plot ($n = 3$):

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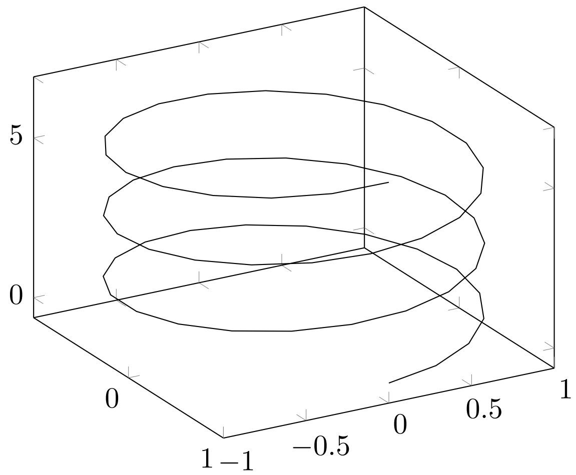
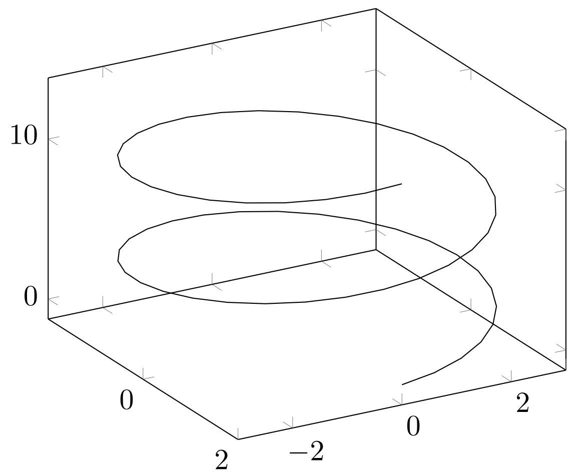


(b) Let $a, b > 0$. Then we have

$$
\begin{aligned}
I_{\beta_{a,b}} &= \frac{1}{2\pi} \int_0^{4\pi} \frac{ab(\sin(t)^2 + \cos(t)^2)}{a^2 \sin(t)^2 + b^2 \cos(t)^2} \, \mathrm{d}t \\
&= \frac{ab}{2\pi} \int_0^{4\pi} \frac{1}{a^2 \sin(t)^2 + b^2 \cos(t)^2} \, \mathrm{d}t \\
&= \frac{4ab}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin(t)^2 + b^2 \cos(t)^2} \, \mathrm{d}t \\
&= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{(\frac{a}{b} \tan(t))^2 + 1} \frac{\frac{a}{b}}{\cos(t)^2} \, \mathrm{d}t \\
&= \frac{4}{\pi} \left[\arctan\left(\frac{a}{b} \tan(t) \right) \right]_0^{\frac{\pi}{2}} \\
&= \frac{4}{\pi} \left(\frac{\pi}{2} - 0 \right) \\
&= 2.
\end{aligned}
$$

Plot ($a = 2, b = 3$):



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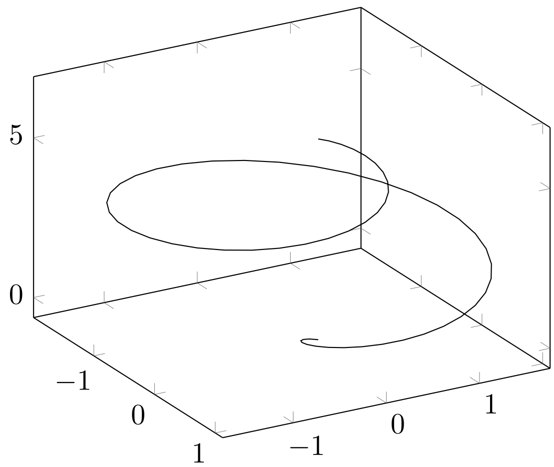
(c) We have

$$\begin{aligned}
I_\gamma &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(-\sin(t)+2\sin(2t))(-\sin(t)+4\sin(2t)) - (-\cos(t)+4\cos(2t))(\cos(t)-2\cos(2t))}{(-\sin(t)+2\sin(2t))^2+(\cos(t)-2\cos(2t))^2}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{9-6\cos(t)}{5-4\cos(t)}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{9-6\cos(t+\pi)}{5-4\cos(t+\pi)}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{9+6\cos(t)}{5+4\cos(t)}\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{0}^{\pi} \frac{9+6\cos(t)}{5+4\cos(t)}\,\mathrm{d}t \\
&= \ldots \\
&= \frac{1}{\pi}\left[\frac{3}{2}t - \arctan\left(3\cot\left(\frac{t}{2}\right)\right)\right]_0^\pi \\
&= \frac{1}{\pi}\left(\left(\frac{3}{2}\pi - 0\right) - \left(0 - \frac{1}{2}\pi\right)\right) \\
&= 2.
\end{aligned}$$

Plot:



Exercise 2.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^3$ be a regular paramterized curve with nowhere vanishing curvature κ and nowhere vanishing torsion τ. Show the equivalence of the following statements:

(i) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ such that $t \cdot v$ is constant.

(ii) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with $n \cdot v \equiv 0$.

(iii) There exists a vector $v \in \mathbb{R}^3 \setminus \{0\}$ such that $b \cdot v$ is constant.

(iv) The ratio of the torsion τ and the curvature κ is constant.

A curve satisfying one of these equivalent conditions is called a *generalized helix*.

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Solution 2.
Without loss of generality, α is parameterized by arc length.

Let $v \in \mathbb{R}^3 \setminus \{0\}$. Then we have

$$\frac{1}{\kappa}(t \cdot v)' = \frac{1}{\kappa}(t' \cdot v + t \cdot v') = \frac{1}{\kappa}(\alpha'' \cdot v) = n \cdot v$$

and hence

$$\frac{\tau}{\kappa}(t \cdot v)' = \frac{\tau}{\kappa}(t' \cdot v + t \cdot v') = \frac{\tau}{\kappa}(\alpha'' \cdot v) = \tau(n \cdot v) = (b' \cdot v) = (b \cdot v)'.$$

The implications (i) $\iff$ (ii) $\iff$ (iii) are now clear.

(i) $\implies$ (iv): Let $\lambda = t \cdot v$. By the second identity, there exists $\mu \in \mathbb{R}$ with $\mu = b \cdot v$ and, by the first identity, we have $n \cdot v = 0$. Since (t, n, b) is an orthonormal basis, we conclude that

$$v = (t \cdot v)\ t + (n \cdot v)\ n + (b \cdot v)\ b = \lambda t + \mu b.$$

Hence, with the Frenet formulas, we obtain

$$0 = \lambda t' + \mu b' = \lambda \kappa n + \mu \tau n = \kappa \left(\lambda + \mu \frac{\tau}{\kappa} \right) n,$$

and therefore

$$\lambda + \mu \frac{\tau}{\kappa} = 0.$$

Assume that $\mu = 0$. Then $v = \lambda t \neq 0$ and therefore $t = 1/\lambda v$. But this contradicts $\kappa \neq 0$. Hence, we condlude that

$$\frac{\tau}{\kappa} = -\frac{\lambda}{\mu} = \text{const},$$

i.e. (iv) holds.

(iv) $\implies$ (ii): Set $v = \frac{\tau}{\kappa} t - b$. Then $v \neq 0$, since t and b are linear independent. With the Frenet formulas we obtain

$$v' = \frac{\tau}{\kappa} t' - b' = \frac{\tau}{\kappa} \kappa n - \tau n = 0,$$

hence v is constant. Therefore it follows that

$$n \cdot v = n \cdot \left(\frac{\tau}{\kappa} t - b \right) = \frac{\tau}{\kappa} n \cdot t - n \cdot b = 0,$$

since (t, n, b) (pointwise) is an orthonormal basis.

Exercise 3.
Consider the function

$$c \colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{cases} (t, e^{-1/t^2}, 0), & \text{falls } t < 0, \\ (0, 0, 0), & \text{falls } t = 0, \\ (t, 0, e^{-1/t^2}), & \text{falls } t > 0. \end{cases}$$

(i) Show that c is a regular, two times continuously differentiable ($c \in C^2(\mathbb{R}, \mathbb{R}^3)$) curve.

(ii) Show that the curvature κ of c only vanishes on $\left\{0, \pm\sqrt{\frac{2}{3}}\right\}$. What does $\kappa(0) = 0$ means graphically?

(iii) Show that the limit of the osculating plane of c at $t \downarrow 0$ is the plane $\{(x, y, z)\ ;\ y = 0\}$, whereas at $t \uparrow 0$ the plane $\{(x, y, z)\ ;\ z = 0\}$ is approximated. What does that mean for the torsion?

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Solution 3.

(i) At first we observe that

$$\lim_{h\to 0} \frac{1}{h^k} e^{-1/h^2} = 0 \tag{1}$$

for all $k \in \mathbb{N}$. Hence

$$c'(0) = \lim_{h\to 0} \frac{c(h) - c(0)}{h} = (1, 0, 0).$$

Furthermore, we obtain

$$c'(t) = \left(1, \frac{2}{t^3} e^{-1/t^2}, 0\right)$$

for $t < 0$ and

$$c'(t) = \left(1, 0, \frac{2}{t^3} e^{-1/t^2}\right)$$

for $t > 0$. Hence, with Eq. (1) it follows that $c \in C^1(\mathbb{R}, \mathbb{R}^3)$. Again with Eq. (1), we conclude that

$$c''(0) = (0, 0, 0)$$

and

$$c''(t) = \left(0, \frac{4 - 6t^2}{t^6} e^{-1/t^2}, 0\right)$$

for all $t < 0$ as well as

$$c''(t) = \left(0, 0, \frac{4 - 6t^2}{t^6} e^{-1/t^2}\right)$$

for all $t > 0$. Overall, again with Eq. (1) we obtain $c \in C^2(\mathbb{R}, \mathbb{R}^2)$.

(ii) With (i) it is immediately clear that the curvature vanishes at 0 and, since $4 - 6t^2 = 0$ if and only if $t \in \left\{\pm\sqrt{\frac{2}{3}}\right\}$, the assertion follows.

The graphs around 0 looks locally like a straight line. If we reparametrize c by arc length, then $\kappa(0) = c''(0) = 0$. Hence, there exists no normal vector in 0 and therefore also no osculating plane in 0.

(iii) Since $|c'| > 0$, it is enough to consider $c' \times c''$. We have

$$c'(0) \times c''(0) = (0, 0, 0)$$

and

$$c'(t) \times c''(t) = \left(0, 0, \frac{4 - 6t^2}{t^6} e^{-1/t^2}\right) = \frac{4 - 6t^2}{t^6} e^{-1/t^2} (0, 0, 1)$$

for all $t < 0$ as well as

$$c'(t) \times c''(t) = \left(0, -\frac{4 - 6t^2}{t^6} e^{-1/t^2}, 0\right) = \frac{4 - 6t^2}{t^6} e^{-1/t^2} (0, -1, 0)$$

for all $t > 0$.

Let $-\sqrt{\frac{2}{3}} \neq t < 0$. The the osculating plane S_t is given by

$$S_t = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;;\; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} t \\ e^{-1/t^2} \\ 0 \end{pmatrix} \right) = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;;\; z = 0 \right\},$$

and hence

$$S_{\uparrow 0} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;;\; z = 0 \right\}.$$

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Now let $\sqrt{\frac{2}{3}} \neq t > 0$. Then the osculating plane S_t is given by

$$S_t = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \; ; \; \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} t \\ 0 \\ e^{-1/t^2} \end{pmatrix} \right) = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \; ; \; y = 0 \right\},$$

and hence

$$S_{\downarrow 0} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \; ; \; y = 0 \right\}.$$

It is not possible to extend the curve in 0 continuously. Since the curves $c|_{-\infty,0)}$ and $c|_{(0,\infty)}$ are planar, we have $\tau = 0$ there. If $\tau = 0$ everywhere, then c would be a planar curve (contradiction).

Exercise 4.
Let $L > 0$ and let $\alpha \colon [0, L] \to \mathbb{R}^2$ be a planar, paramterized by arc length, simple closed curve. The curvature κ satisfies $0 < \kappa(s) \leq c$ for all $s \in [0, L]$ with a constant c. Show: The length L of the curve satisfies

$$L \geq \frac{2\pi}{c}.$$

What does that mean graphically?

Solution 4.
Since $\kappa > 0$, by the theorem of turning tangents we have

$$1 = \frac{1}{2\pi} \int_0^L \kappa(s) \, \mathrm{d}s \leq \frac{c}{2\pi} \int_0^L 1 \, \mathrm{d}s = \frac{Lc}{2\pi}.$$

The shortest possible curve is a circle with radius $1/c$.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.