UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 5 **Submission:** /

Resources: Up to Lektion 10; Up to p. 44 in [Fuc08]; Sections 1-1 – 1-7 B and Section 5-7 up to Proposition 1 in [Car16]

Exercise 1.
Let $I \subset \mathbb{R}$ be an interval, let $\alpha\colon I \to \mathbb{R}^2$ be a regular, planar curve which is parameterized by arc length and let, for $r > 0$,

$$\alpha_r\colon I \to \mathbb{R}^2,\ t \mapsto \alpha(t) \pm r n_\alpha(t),$$

where $n_\alpha\colon I \to \mathbb{R}^2$ is the normal of α. α_r is called *inner* (+) resp. *outer* (−) *parallel curve* to α with distance r.

(i) When is α_r regular? When is α_r parameterized by arc length?

(ii) In the case that α_r is regular, describe the oriented curvature κ_{α_r} of α_r with the oriented curvature κ_α of α.

(iii) Let $I = \mathbb{R}$ and let α be periodic with period $l \in (0, \infty)$. Show that:

$$\frac{\mathrm{d}}{\mathrm{d}r} L(\alpha_r|_{[0,l]})|_{r=0} = \mp 2\pi I(\alpha|_{[0,l]}),$$

where $I(\alpha|_{[0,l]})$ is the rotation index of $\alpha|_{[0,l]}$ and L is the arc length.

Exercise 2.
Let $L \in \mathbb{R}$ and $I = [0, L] \subset \mathbb{R}$. Let α be an *oval*, i.e., a simple closed, regular, parametrized by arc length, and convex curve $\alpha \in C^2(I, \mathbb{R}^2)$ with nowhere vanishing curvature.

(i) Show that for each unit vector e there exists a unique parameter $s \in I$ with $t_\alpha(s) = e$.

(ii) Show that α can be reparametrized with respect to the oriented angle $\vartheta\colon I \to [0, 2\pi]$ between the tangent vector t_α and the x axis. These coordinates are called *tangential polar coordinates*.

(iii) Let β be the reparametrization of the oval α in tangential polar coordinates. The curve β is called a *curve of constant width* if the function $h\colon [0, 2\pi] \to \mathbb{R},\ \vartheta \mapsto -\beta(\vartheta) \cdot n_\beta(\vartheta)$ satisfies the following condition with a constant $d > 0$:

$$h(\vartheta) + h(\vartheta + \pi) = d$$

for all $\vartheta \in [0, \pi]$. Show that a curve with constant width d has a circumference of πd.

(*Hint: Describe β with respect to (n_β, n'_β) and with the help of h.*)

(please turn the page)

Exercise 3.
Let $L > 0$, $\alpha\colon [0, L] \to \mathbb{R}^2$ be a simple closed, convex curve which is parameterized by arc length and is positive oriented, and let α_r be the outer parallel curve with distance $r > 0$ (see Exercise 1). Show that:

(i) $U(\alpha_r) = U(\alpha) + 2\pi r$,

(ii) $A(\alpha_r) = A(\alpha) + Lr + \pi r^2$.

Here, $U(\alpha)$ is the circumference and $A(\alpha)$ is the area enclosed by the curve α.

(*Hint: You can use the following statement without a proof: Let* $a, b \in \mathbb{R}, a < b$ *and let* $\alpha\colon [a, b] \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ *be an injective, continuously differentiable curve with positive curvature. Let* $A = \alpha(a)$ *and* $B = \alpha(b)$. *Then the trace of* α *and the line segments* $\overline{OA}$ *and* $\overline{BO}$ *enclose a bounded domain* $S \subset \mathbb{R}^2$ *whose area can be calculated via the formula*

$$A(S) = \frac{1}{2}\int_a^b x(t)y'(t) - x'(t)y(t)\,\mathrm{d}t.)$$

Exercise 4.
Let $a > 0$ and

$$r\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R},\ t \mapsto a\frac{\cos(2t)}{\cos(t)}.$$

Consider the following planar curve which is given in polar coordinates (a *strophoid*):

$$\alpha\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2,\ t \mapsto (r(t)\cos(t), r(t)\sin(t)).$$

(i) Calculate the intersection points of the curve with the axes and show that the straight line $\{(x, y) \in \mathbb{R}^2\ ;\ x = -a\}$ is the asymptote of the curve.

(ii) For the curve α, there exist $t_1 \neq t_2$ with $\alpha(t_1) = \alpha(t_2) = 0$, hence the curve has a loop there. Show that the area enclosed by this loop is given by $\left(2 - \frac{\pi}{2}\right) a^2$ (Plot!).

(iii) The curve and its asmyptote encloses an area which extends into infinity. Show that the area is given by $\left(2 + \frac{\pi}{2}\right) a^2$.

(*Hint: Consider the curve which is translated by the vector* $(a, 0)$ *and use the formula from Exercise 3.*)

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.