UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 5, Solution **Submission:** /

Resources: Up to Lektion 10; Up to p. 44 in [Fuc08]; Sections 1-1 – 1-7 B and Section 5-7 up to Proposition 1 in [Car16]

Exercise 1.
Let $I \subset \mathbb{R}$ be an interval, let $\alpha\colon I \to \mathbb{R}^2$ be a regular, planar curve which is parameterized by arc length and let, for $r > 0$,

$$\alpha_r\colon I \to \mathbb{R}^2,\ t \mapsto \alpha(t) \pm r n_\alpha(t),$$

where $n_\alpha\colon I \to \mathbb{R}^2$ is the normal of α. α_r is called *inner* (+) resp. *outer* (−) *parallel curve* to α with distance r.

(i) When is α_r regular? When is α_r parameterized by arc length?

(ii) In the case that α_r is regular, describe the oriented curvature κ_{α_r} of α_r with the oriented curvature κ_α of α.

(iii) Let $I = \mathbb{R}$ and let α be periodic with period $l \in (0, \infty)$. Show that:

$$\frac{\mathrm{d}}{\mathrm{d}r} L(\alpha_r|_{[0,l]})|_{r=0} = \mp 2\pi I(\alpha|_{[0,l]}),$$

where $I(\alpha|_{[0,l]})$ is the rotation index of $\alpha|_{[0,l]}$ and L is the arc length.

Solution 1.

(i) We have

$$\alpha_r' = \alpha' \pm r n_\alpha' = \alpha' \mp r\kappa_\alpha t_\alpha = \alpha' \mp r\kappa_\alpha \alpha' = (1 \mp r\kappa_\alpha)\alpha',$$

hence

$$|\alpha_r'|^2 = |1 \mp r\kappa_\alpha|^2 |\alpha'|^2 = |1 \mp r\kappa_\alpha|^2.$$

Then α_r is regular if and only if $1 \mp r\kappa_\alpha \neq 0$ on I. Furthermore, α_r is parameterized by arc length if and only if

$$\kappa_\alpha = 0 \quad \text{or} \quad \kappa_\alpha = \pm\frac{2}{r}.$$

(ii) Let $\alpha\colon I \to \mathbb{R}^2,\ t \mapsto (x(t), y(t))$ and let $\alpha_r\colon I \to \mathbb{R}^2,\ t \mapsto (x_r(t), y_r(t))$ be regular. With

$$\alpha_r'' = (\mp r\kappa_\alpha')\alpha' + (1 \mp r\kappa_\alpha)\alpha''$$

and Sheet 2, Exercise 4 (i) we conclude that

$$\begin{aligned}
\kappa_{\alpha_r} &= \frac{x_r' y_r'' - x_r'' y_r'}{|\alpha_r'|^3} \\
&= \frac{(1 \mp r\kappa_\alpha)x'((\mp r\kappa_\alpha')y' + (1 \mp r\kappa_\alpha)y'') - ((\mp r\kappa_\alpha')x' + (1 \mp r\kappa_\alpha)x'')(1 \mp r\kappa_\alpha)y'}{|1 \mp r\kappa_\alpha|^3} \\
&= \frac{(1 \mp r\kappa_\alpha)^2 x'y'' + (1 \mp r\kappa_\alpha)(\mp r\kappa_\alpha')x'y' - (1 \mp r\kappa_\alpha)^2 x''y' - (\mp r\kappa_\alpha')(1 \mp r\kappa_\alpha)x'y'}{|1 \mp r\kappa_\alpha|^3} \\
&= \frac{\kappa_\alpha}{|1 \mp r\kappa_\alpha|}.
\end{aligned}$$

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(iii) Since α is periodic, κ_α is bounded. Hence, for r small enough, we have $1 \mp r\kappa_\alpha > 0$. With

$$L(\alpha_r|_{[0,l]}) = \int_0^l |\alpha_r'(t)| \, \mathrm{d}t = \int_0^l |1 \mp r\kappa_\alpha(t)| \, \mathrm{d}t$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}|_{r=0} |1 \mp r\kappa_\alpha| = \left(\frac{(1 \mp r\kappa_\alpha)(\mp\kappa_\alpha)}{|1 \mp r\kappa_\alpha|} \right) |_{r=0} = \mp\kappa_\alpha$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}|_{r=0} L(\alpha_r|_{[0,l]}) = \int_0^l \frac{\mathrm{d}}{\mathrm{d}r}|_{r=0} |1 \mp r\kappa_\alpha(t)| \, \mathrm{d}t = \mp \int_0^l \kappa_\alpha(t) \, \mathrm{d}t = \mp 2\pi I(\alpha|_{[0,l]}).$$

Exercise 2.

Let $L \in \mathbb{R}$ and $I = [0, L] \subset \mathbb{R}$. Let α be an *oval*, i.e., a simple closed, regular, parametrized by arc length, and convex curve $\alpha \in C^2(I, \mathbb{R}^2)$ with nowhere vanishing curvature.

(i) Show that for each unit vector e there exists a unique parameter $s \in I$ with $t_\alpha(s) = e$.

(ii) Show that α can be reparametrized with respect to the oriented angle $\vartheta \colon I \to [0, 2\pi]$ between the tangent vector t_α and the x axis. These coordinates are called *tangential polar coordinates.*

(iii) Let β be the reparametrization of the oval α in tangential polar coordinates. The curve β is called a *curve of constant width* if the function $h \colon [0, 2\pi] \to \mathbb{R}$, $\vartheta \mapsto -\beta(\vartheta) \cdot n_\beta(\vartheta)$ satisfies the following condition with a constant $d > 0$:

$$h(\vartheta) + h(\vartheta + \pi) = d$$

for all $\vartheta \in [0, \pi]$. Show that a curve with constant width d has a circumference of πd.

(*Hint: Describe* β *with respect to* (n_β, n_β') *and with the help of* h.)

Solution 2.

(i) Without loss of generality, we assume that $\kappa > 0$. Then $\kappa = \vartheta' > 0$, thus ϑ is strictly increasing. Without loss of generality, we can assume that $\vartheta(0) = 0$ and, since $I_\alpha = 1$ by the theorem of turning tangents, we obtain $\vartheta(L) = 2\pi$. Since $\vartheta([0, L]) = [0, 2\pi]$, $\vartheta \colon [0, L] \to [0, 2\pi]$ is bijective. Since every unit vector is uniquely determined by its angle with the x axis, the result follows.

(ii) Since ϑ is bijective and $\vartheta' = \kappa \neq 0$ on I, it follows that ϑ^{-1} is differentiable, hence ϑ is a diffeomorphism. Thus a reparametrization is possible.

(iii) Let $\beta \colon [0, 2\pi] \to \mathbb{R}^2$ be the reparametrization of α with respect to ϑ and $h \colon [0, 2\pi] \to \mathbb{R}$, $\vartheta \mapsto -\beta(\vartheta) \cdot n_\beta(\vartheta)$. Since

$$t_\beta = (\cos, \sin)$$

by construction, it follows that

$$n_\beta = (-\sin, \cos), \quad n_\beta' = (-\cos, -\sin) = -t_\beta \quad \text{and} \quad n_\beta'' = (\sin, -\cos) = -n_\beta.$$

Hence (n_β, n_β') is (pointwise) an orthonormal basis of $\mathbb{R}^2$. Furthermore, we have

$$h' = -\beta' \cdot n_\beta - \beta \cdot n_\beta' = -t_\beta \cdot n_\beta - \beta \cdot n_\beta' = -\beta \cdot n_\beta',$$

thus

$$\beta = (\beta \cdot n_\beta) n_\beta + (\beta \cdot n_\beta') n_\beta' = -h n_\beta - h' n_\beta'$$

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and therefore

$$\beta' = -h'n_\beta - hn'_\beta - h''n'_\beta - h'n''_\beta = -h'n_\beta - (h+h'')n'_\beta + h'n_\beta = -(h+h'')n'_\beta = (h+h'')t_\beta.$$

Since

$$h' = -\beta \cdot n'_\beta \quad \text{and} \quad h'' = -\beta' \cdot n'_\beta - \beta \cdot n''_\beta = \beta \cdot n_\beta - \beta' \cdot n'_\beta,$$

we obtain

$$h'' + h = \beta \cdot n_\beta - \beta' \cdot n'_\beta - \beta \cdot n_\beta = -\beta' \cdot n'_\beta = -|\beta'|(t_\beta \cdot n'_\beta) = |\beta'|.$$

Finally, from $h(\vartheta) + h(\vartheta + \pi) = d$ for all $\vartheta \in [0, \pi]$ we conlude that

$$h'(\vartheta) + h'(\vartheta + \pi) = 0$$

for all $\vartheta \in [0, \pi]$ and hence

$$h'(0) + h'(\pi) = 0 \quad \text{as well as} \quad h'(\pi) + h'(2\pi) = 0.$$

Thus

$$h'(2\pi) - h'(0) = 0.$$

The circumference is now given by

$$\begin{aligned}
U(\beta) &= \int_0^{2\pi} |\beta'| = \int_0^{2\pi} h + \int_0^{2\pi} h'' \\
&= \left(\int_0^{\pi} h + \int_\pi^{2\pi} h \right) + (h'(2\pi) - h'(0)) \\
&= \int_0^{\pi} h + \int_0^{\pi} h(\vartheta + \pi)\,\mathrm{d}\vartheta \\
&= \int_0^{\pi} h(\vartheta) + h(\vartheta + \pi)\,\mathrm{d}\vartheta \\
&= \int_0^{\pi} d \\
&= \pi d.
\end{aligned}$$

Exercise 3.
Let $L > 0$, $\alpha\colon [0, L] \to \mathbb{R}^2$ be a simple closed, convex curve which is parameterized by arc length and is positive oriented, and let α_r be the outer parallel curve with distance $r > 0$ (see Exercise 1). Show that:

(i) $U(\alpha_r) = U(\alpha) + 2\pi r$,

(ii) $A(\alpha_r) = A(\alpha) + Lr + \pi r^2$.

Here, $U(\alpha)$ is the circumference and $A(\alpha)$ is the area enclosed by the curve α.

(*Hint: You can use the following statement without a proof: Let $a, b \in \mathbb{R}, a < b$ and let $\alpha\colon [a, b] \to \mathbb{R}^2$, $t \mapsto (x(t), y(t))$ be an injective, continuously differentiable curve with positive curvature. Let $A = \alpha(a)$ and $B = \alpha(b)$. Then the trace of α and the line segments $\overline{OA}$ and $\overline{BO}$ enclose a bounded domain $S \subset \mathbb{R}^2$ whose area can be calculated via the formula*

$$A(S) = \frac{1}{2} \int_a^b x(t)y'(t) - x'(t)y(t)\,\mathrm{d}t.)$$

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Solution 3.

(i) We have

$$U(\alpha_r) = \int_0^L |\alpha_r'| = \int_0^L |1 + r\kappa_\alpha| = \int_0^L (1 + r\kappa_\alpha) = L + r\int_0^L \kappa_\alpha = L + 2\pi r.$$

(ii) We have

$$\begin{aligned}
2A(\alpha_r) &= \int_0^L x_r y_r' - x_r' y_r \\
&= \int_0^L (x - r(-y'))(1 + r\kappa_\alpha)y' - (1 + r\kappa_\alpha)x'(y - rx') \\
&= \int_0^L xy'(1 + r\kappa_\alpha) + ry'^2(1 + r\kappa_\alpha) - (1 + r\kappa_\alpha)x'y + r(1 + r\kappa_\alpha)x'^2 \\
&= \int_0^L (1 + r\kappa_\alpha)(xy' - x'y) + r(1 + r\kappa_\alpha)(x'^2 + y'^2) \\
&= \int_0^L (xy' - x'y) + r\int_0^L \kappa_\alpha(xy' - x'y) + r\int_0^L 1 + r^2\int_0^L \kappa_\alpha \\
&= 2A(\alpha) + rL + 2\pi r^2 + r\int_0^L \kappa_\alpha(xy' - x'y).
\end{aligned}$$

Since

$$x'^2 + y'^2 = 1$$

and therefore

$$x'x'' + y'y'' = 0,$$

we obtain that

$$\begin{aligned}
\kappa_\alpha(xy' - x'y) &= (x'y'' - x''y')(xy' - x'y) \\
&= x'xy'y'' - x'^2yy'' - xx''y'^2 + x'x''yy' \\
&= -xx'^2x'' - x'^2yy'' - xx''y'^2 - yy'^2y'' \\
&= -(xx''(x'^2 + y'^2) + yy''(x'^2 + y'^2)) \\
&= -(xx'' + yy'')
\end{aligned}$$

and

$$\begin{aligned}
-\int_0^L (xx'' + yy'') &= -\left([xx']_0^L - \int_0^L x'^2 + [yy']_0^L - \int_0^L y'^2\right) \\
&= \int_0^L x'^2 + y'^2 - [\alpha \cdot \alpha']_0^L \\
&= L.
\end{aligned}$$

Since α is closed, the result follows.

Exercise 4.

Let $a > 0$ and

$$r\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R},\ t \mapsto a\frac{\cos(2t)}{\cos(t)}.$$

Consider the following planar curve which is given in polar coordinates (a *strophoid*):

$$\alpha\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2,\ t \mapsto (r(t)\cos(t), r(t)\sin(t)).$$

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(i) Calculate the intersection points of the curve with the axes and show that the straight line $\{(x,y) \in \mathbb{R}^2 \ ; \ x = -a\}$ is the asymptote of the curve.

(ii) For the curve α, there exist $t_1 \neq t_2$ with $\alpha(t_1) = \alpha(t_2) = 0$, hence the curve has a loop there. Show that the area enclosed by this loop is given by $\left(2 - \frac{\pi}{2}\right) a^2$ (Plot!).

(iii) The curve and its asmyptote encloses an area which extends into infinity. Show that the area is given by $\left(2 + \frac{\pi}{2}\right) a^2$.

(*Hint: Consider the curve which is translated by the vector* $(a, 0)$ *and use the formula from Exercise 3.*)

Solution 4.
We have

$$\alpha(t) = a(\cos(2t), \cos(2t)\tan(t))$$

for all $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(i) We have

$$\cos(2t) = 0 \iff t = \pm\frac{\pi}{4}$$

and

$$\cos(2t)\tan(t) = 0 \iff t = \pm\frac{\pi}{4} \text{ or } t = 0.$$

Hence α intersects the x axis in $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ and the y axis in $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$. Furthermore, we have

$$\lim_{t\to-\frac{\pi}{2}} \cos(2t) = -1 \quad \text{and} \quad \lim_{t\to-\frac{\pi}{2}} \cos(2t)\tan(t) = \infty$$

and

$$\lim_{t\to\frac{\pi}{2}} \cos(2t) = -1 \quad \text{and} \quad \lim_{t\to\frac{\pi}{2}} \cos(2t)\tan(t) = -\infty.$$

Thus the straight line $\{(x,y) \in \mathbb{R}^2 \ ; \ x = -a\}$ is the asymptote of α.

(ii) We first consider the lower half. Let $\frac{\pi}{4} > \varepsilon > 0$ and $\alpha_\varepsilon \colon \left[-\frac{\pi}{4} + \varepsilon, 0\right] \to \mathbb{R}^2$, $t \mapsto \alpha(t)$. Then we obtain (see the hint of Exercise 3)

$$\begin{aligned}
A(S_\varepsilon) &= \frac{1}{2}a^2 \int_{-\frac{\pi}{4}+\varepsilon}^{0} \cos(2t)\left(-2\sin(2t)\tan(t) + \cos(2t)\frac{1}{\cos(t)^2}\right) \\
&\qquad - (-2\sin(2t))\cos(2t)\tan(t)\,\mathrm{d}t \\
&= \frac{1}{2}a^2 \int_{-\frac{\pi}{4}+\varepsilon}^{0} \left(\frac{\cos(2t)}{\cos(t)}\right)^2 \mathrm{d}t \\
&= \ldots \\
&= \frac{1}{2}a^2[-2t + \sin(2t) + \tan(t)]_{-\frac{\pi}{4}+\varepsilon}^{0} \\
&= \frac{1}{2}a^2\left(0 - \left(\frac{\pi}{2} - 2\varepsilon + \sin\left(-\frac{\pi}{2} + 2\varepsilon\right) + \tan\left(-\frac{\pi}{4} + \varepsilon\right)\right)\right) \\
&\to \frac{1}{2}a^2\left(-\frac{\pi}{2} + 2\right).
\end{aligned}$$

for $\varepsilon \to 0$. The result follows.

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(iii) We first consider the upper half. Let $\frac{\pi}{4} > \varepsilon > 0$ and $\alpha_\varepsilon \colon \left[-\frac{\pi}{2} + \varepsilon, -\frac{\pi}{4}\right] \to \mathbb{R}^2$, $t \mapsto \alpha(t) + (a, 0)$. Then we obtain (see the hint of Exercise 3)

$$\begin{aligned} A(S_\varepsilon) &= \frac{1}{2}a^2 \int_{-\frac{\pi}{2}+\varepsilon}^{-\frac{\pi}{4}} (\cos(2t) + 1) \left(-2\sin(2t)\tan(t) + \cos(2t)\frac{1}{\cos(t)^2}\right) \\ &\qquad - (-2\sin(2t))\cos(2t)\tan(t)\,\mathrm{d}t \\ &= \frac{1}{2}a^2 \int_{-\frac{\pi}{2}+\varepsilon}^{-\frac{\pi}{4}} \left(\frac{\cos(2t)}{\cos(t)}\right)^2 - 2\sin(2t)\tan(t) + \frac{\cos(2t)}{\cos(t)^2}\,\mathrm{d}t \\ &= \dots \\ &= \frac{1}{2}a^2[-2t + 2\sin(2t)]_{-\frac{\pi}{2}+\varepsilon}^{-\frac{\pi}{4}} \\ &= \frac{1}{2}a^2 \left(\left(\frac{\pi}{2} - 2\right) - (\pi - 2\varepsilon + 2\sin(-\pi + 2\varepsilon))\right) \\ &= \frac{1}{2}a^2 \left(-\frac{\pi}{2} - 2 - 2\varepsilon - 2\sin(-\pi + 2\varepsilon)\right) \\ &\to -\frac{1}{2}a^2 \left(\frac{\pi}{2} + 2\right) \end{aligned}$$

for $\varepsilon \to 0$. The result follows.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.