UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 6 **Submission:** /

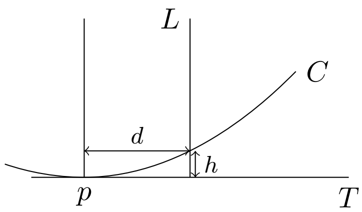
Resources: Up to Lektion 10; Up to p. 44 in [Fuc08]; Sections 1-1 – 1-7 B and Section 5-7 up to Proposition 1 in [Car16]

Exercise 1.

(i) Let C be a planar curve, T the tangent of C in $p \in C$ and let L be a straight line parallel to the normal in p with distance d to p (see below). Let h be the length of segement of the L which is determined by C and T (h is the "height" of C relative to T). Show that

$$|\kappa(p)| = \lim_{d \to 0} \frac{2h}{d}$$

holds.



(ii) Show: If a closed, planar curve C is contained in a circle with radius r, then there exists a point $p \in C$ such that the curvature κ of C in p satisfies

$$|\kappa| \geq \frac{1}{r}.$$

Exercise 2.
Let $\alpha \colon \mathbb{R} \to \mathbb{R}^2$ be a simple closed, planar curve which is parameterized by arc length and denote by $\kappa \colon \mathbb{R} \to \mathbb{R}$ the (oriented) curvature of this curve. Show that α is convex if $\kappa(s) \geq 0$ for all $s \in \mathbb{R}$ or $\kappa(s) \leq 0$ for all $s \in \mathbb{R}$.

Exercise 3.

(i) Does a simple closed, planar curve with a length of $6m$ and an enclosed area of $4m^2$ exist? Justify your answer.

(ii) Let $\overline{AB}$ be a line segment in $\mathbb{R}^2$ and let $l > |\overline{AB}|$. Proof that a curve with length l which connects the points A and B and maximizes the area enclosed by the curve and the line segment $\overline{AB}$ is an arc of a circle passing through A and B.

(please turn the page)

Exercise 4.
Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^3$ be a curve which is parameterized by arc length and has the curvature κ and the torsion τ. Let $\kappa \neq 0$, $\kappa' \neq 0$ and $\tau \neq 0$ on I. The functions κ and τ satisfy the equation

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right) = r^2$$

on I, where $r > 0$ is a constant. Show that α lies on a sphere with radius r.

(*Hint: Consider the curve*

$$\beta\colon I \to \mathbb{R}^3, \; s \mapsto \alpha(s) + \frac{1}{\kappa(s)}n(s) + \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}b(s)$$

where (t, n, b) *is the Frenet trihedron of* α.)

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.