UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 6, Solution **Submission:** /

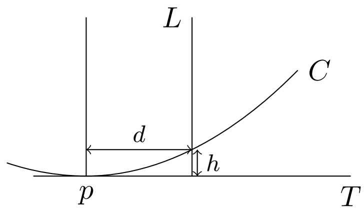
Resources: Up to Lesson 10; Up to p. 44 in [Fuc08]; Sections 1-1 – 1-7 B and Section 5-7 up to Proposition 1 in [Car16]

Exercise 1.

(i) Let C be a planar curve, T the tangent of C in $p \in C$ and let L be a straight line parallel to the normal in p with distance d to p (see below). Let h be the length of segement of the L which is determined by C and T (h is the "height" of C relative to T). Show that

$$|\kappa(p)| = \lim_{d \to 0} \frac{2h}{d}$$

holds.



(ii) Show: If a closed, planar curve C is contained in a circle with radius r, then there exists a point $p \in C$ such that the curvature κ of C in p satisfies

$$|\kappa| \geq \frac{1}{r}.$$

Solution 1.

(i) Without loss of generality, we assume that $p = (0,0)$, T lies on the x axis (i.e. $\alpha'(0) = \big(1,0\big)$) and the normal of C in p lies on the y axis. Furthermore, let $\alpha = (x,y)$ be parameterized by arc length and $p = \alpha(0)$.

Consider the Taylor expansion at 0:

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \frac{1}{2}\alpha''(0)s^2 + R$$

with $R = (R_x, R_y)$ and $\lim_{s\to 0} \frac{R}{s^2} = 0$. Let κ be the curvature of α at $s = 0$. By the Frenet formulas (and the choice of the coordinates), we have

$$\alpha''(0) = \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and hence

$$x(s) = s + R_x, \quad y(s) = \frac{\kappa}{2}s^2 + R_y,$$

Thus

$$|\kappa(p)| = \lim_{s\to 0} \frac{2|y(s)|}{s^2} = \lim_{d\to 0} \frac{2h}{d^2}.$$

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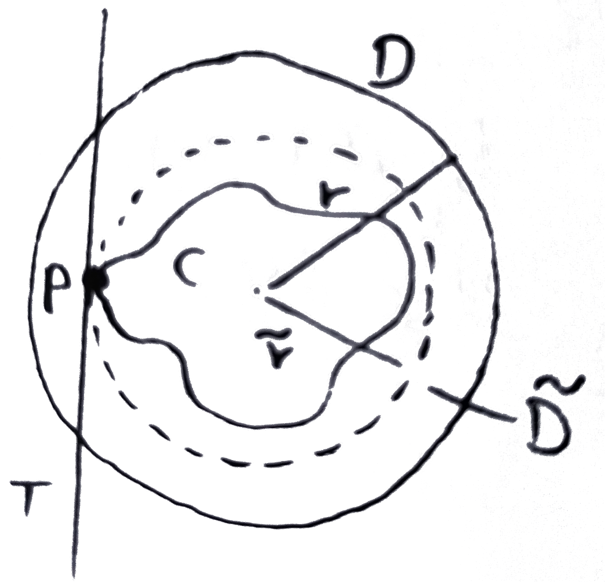

Figure 1: Sketch for Exercise 1 (ii)

(ii) (See Figure 1) Let 0 be the center of the circle D. Reduce the boundary of D by a family of concentric circles until it hits the curve C in a point p. Let T be the joint tangent on $\tilde{D}$ and C in p. As in (i), let $p = (0,0)$ and T lies on the x axis, C is parameterized by $\alpha = (x, y)$, $\alpha(0) = p$, and $\tilde{D}$ is parameterized by $\tilde{\alpha} = (\tilde{x}, \tilde{y})$, $\tilde{\alpha}(0) = p$ (without loss of generality parameterized by arc length). Then, in a small neighborhood of 0, we have

$$\tilde{y}(s) \leq y(s)$$

and with (i) we conclude that

$$\frac{1}{r} \leq \frac{1}{\tilde{r}} = \lim_{s \to 0} \frac{2|\tilde{y}(s)|}{s^2} \leq \lim_{s \to 0} \frac{2|y(s)|}{s^2} = |\kappa(p)|.$$

Exercise 2.

Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ be a simple closed, planar curve which is parameterized by arc length and denote by $\kappa\colon \mathbb{R} \to \mathbb{R}$ the (oriented) curvature of this curve. Show that α is convex if $\kappa(s) \geq 0$ for all $s \in \mathbb{R}$ or $\kappa(s) \leq 0$ for all $s \in \mathbb{R}$.

Solution 2.

Without loss of generality, let $\kappa \geq 0$. We assume that α is not convex. Then there exists $s_0 \in \mathbb{R}$ such that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R},\ s \mapsto (\alpha(s) - \alpha(s_0)) \cdot n(s_0)$$

produces negative as well as positive values. Since α is periodic, φ obtains its minimum in a point $s_1 \in \mathbb{R}$ and the maximum in a point $s_2 \in \mathbb{R}$. Therefore

$$\varphi(s_1) < \varphi(s_0) < \varphi(s_2). \tag{1}$$

Since there is an extremum in s_1, we have $\varphi'(s_1) = 0$. Thus $\alpha'(s_1) \cdot n(s_0) = 0$ and hence $\alpha'(s_1) = \pm\alpha'(s_0)$. Similarly, there is an extremum in s_2 and hence $\alpha'(s_2) = \pm\alpha'(s_0)$. Therefore, two of the three unit vectors $\alpha'(s_0), \alpha'(s_1), \alpha'(s_2)$ have to coincide. We choose $\tilde{s}_1, \tilde{s}_2 \in \{s_0, s_1, s_2\}$ with $\tilde{s}_1 < \tilde{s}_2$ such that $\alpha'(\tilde{s}_1) = \alpha'(\tilde{s}_2)$.

Now let ϑ be an arc function $\mathbb{R} \to \mathbb{R}$, where $\vartheta' = \kappa$. It follows that

$$\vartheta(\tilde{s}_2) - \vartheta(\tilde{s}_1) = 2\pi k \quad (k \in \mathbb{Z}).$$

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Since $\vartheta' = \kappa$, ϑ is increasing and hence $\vartheta(\tilde{s}_2) - \vartheta(\tilde{s}_1) \geq 0$. Thus $k \in \mathbb{N}_0$. In the same way we obtain

$$\vartheta(\tilde{s}_1 + L) - \vartheta(\tilde{s}_2) = 2\pi l \quad (l \in \mathbb{N}_0).$$

The rotation index I_α satisfies $I_\alpha = k + l \geq 0$. The theorem of turning tangents provides $I_\alpha = 1$, thus $k = 0$ or $l = 0$. Without loss of generality, let $k = 0$. It follows that $\kappa = \vartheta' = 0$ on $[\tilde{s}_1, \tilde{s}_2]$. Thus α parameterizes a straight line on $[\tilde{s}_1, \tilde{s}_2]$, i.e., for all $s \in [\tilde{s}_1, \tilde{s}_2]$, we have

$$\alpha(s) = \alpha(\tilde{s}_1) + (s - \tilde{s}_1)\alpha'(\tilde{s}_1) = \alpha(\tilde{s}_1) \pm (s - \tilde{s}_1)\alpha'(s_0).$$

We conclude that

$$\begin{aligned}\varphi(s) = (\alpha(s) - \alpha(s_0)) \cdot n(s_0) &= (\alpha(\tilde{s}_1) \mp (s - \tilde{s}_1)\alpha'(s_0) - \alpha(s_0)) \cdot n(s_0) \\ &= (\alpha(\tilde{s}_1) - \alpha(s_0)) \cdot n(s_0) \mp (s - \tilde{s}_1)\alpha'(s_0) \cdot n(s_0) \\ &= (\alpha(\tilde{s}_1) - \alpha(s_0)) \cdot n(s_0),\end{aligned}$$

hence φ is constant. Since at least two of the three points s_0, s_1, s_2 are contained in the interval $[\tilde{s}_1, \tilde{s}_2]$, we have a contradiction to Eq. (1).

Exercise 3.

(i) Does a simple closed, planar curve with a length of $6m$ and an enclosed area of $4m^2$ exist? Justify your answer.

(ii) Let $\overline{AB}$ be a line segment in $\mathbb{R}^2$ and let $l > |\overline{AB}|$. Proof that a curve with length l which connects the points A and B and maximizes the area enclosed by the curve and the line segment $\overline{AB}$ is an arc of a circle passing through A and B.

Solution 3.

(i) By the isoperimetric inequality, we have

$$3m^2 \leq \frac{(6m)^2}{4\pi} = \frac{9}{\pi}m^2 < 3m^2.$$

Hence, such a curve is not possible.

(ii) Consider Figure 2. Let $L(\beta) = L(\alpha) = l$. Then we have

$$U_{\beta\cup\gamma} = L(\beta \cup \gamma) = L(\beta) + L(\gamma) = l + (2\pi r - L(\alpha)) = l + 2\pi r - l = 2\pi r = L(\alpha \cup \gamma) = U_{\alpha\cup\gamma}$$

and therefore, by the isoperimetric inequality,

$$A_{\beta\cup\gamma} \leq \frac{U^2_{\beta\cup\gamma}}{4\pi} = \frac{U^2_{\alpha\cup\gamma}}{4\pi} = A_{\alpha\cup\gamma}.$$

Thus α is optimal.

Exercise 4.

Let $I \subset \mathbb{R}$ be an interval and let $\alpha\colon I \to \mathbb{R}^3$ be a curve which is parameterized by arc length and has the curvature κ and the torsion τ. Let $\kappa \neq 0$, $\kappa' \neq 0$ and $\tau \neq 0$ on I. The functions κ and τ satisfy the equation

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right) = r^2$$

on I, where $r > 0$ is a constant. Show that α lies on a sphere with radius r.

(*Hint: Consider the curve*

$$\beta\colon I \to \mathbb{R}^3, \quad s \mapsto \alpha(s) + \frac{1}{\kappa(s)}n(s) + \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}b(s)$$

where (t, n, b) *is the Frenet trihedron of* α.)

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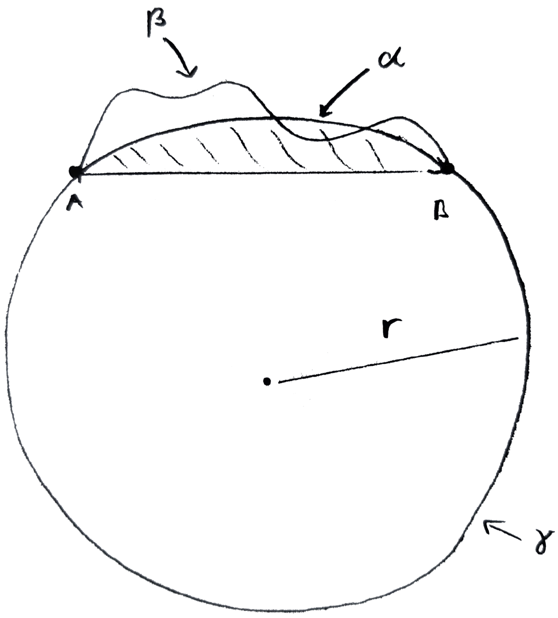

Figure 2: Sketch for Exercise 3 (ii)

Solution 4.
Define $K = \frac{1}{\kappa}$ and $T = \frac{1}{\tau}$. Then

$$K' = -\frac{\kappa'}{\kappa^2},$$

and thus

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right) = r^2 \qquad \Longleftrightarrow \qquad K^2 + (K'T)^2 = r^2.$$

Differentiating the equation on the right, we obtain

$$2KK' + 2(K'T)(K'T)' = \frac{2K'}{\tau}(K\tau + (K'T)') = 0,$$

hence

$$K\tau + (K'T)' = 0$$

on I. Furthermore, by the Frenet formulas, we conclude that

$$\begin{aligned}\beta' &= \alpha' + K'n + Kn' + (K'T)'b + (K'T)b' \\ &= t + K'n + K(\tau b - \kappa t) + (K'T)'b - K'T\tau n \\ &= (1 - K\kappa)t + (K' - K'T\tau)n + (K\tau + (K'T)')b \\ &= 0,\end{aligned}$$

thus β is constant. Since (t, n, b) is (pointwise) an orthonormal basis, we obtain with the definition of β that

$$(\alpha - \beta)^2 = \left(-\frac{1}{\kappa}\right) \cdot \left(-\frac{1}{\kappa}\right) + \left(-\frac{\kappa'}{\kappa^2\tau}\right) \cdot \left(-\frac{\kappa'}{\kappa^2\tau}\right) = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right) = r^2.$$

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.