UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
Prof. Dr. Martin Fuchs
Dr. Dominik Schillo

Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 7, Solution **Submission:** /

Resources: Up to Lesson 13; Up to p. 56 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 143 in [Car16]

Exercise 1.
Consider for $a, b, c > 0$ the subsets of $\mathbb{R}^3$:

(i) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$ (Ellipsoid),

(ii) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\right\}$ (Hyperboloid of one sheet),

(iii) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1\right\}$ (Hyperboloid of two sheets),

(iv) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0\right\}$ (Elliptic paraboloid),

(v) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0\right\}$ (Hyperbolic paraboloid).

Sketch these sets and describe subsets (as large as possible) via parameterized surfaces.

Exercise 2.
Let $I \subset \mathbb{R}$ be an interval, C the trace of an regular, parameterized by arc length, injective (planar) $\alpha \colon I \to \mathbb{R}^3$ with $\operatorname{Im}(\alpha) \subset \{(x, y, z) \in \mathbb{R}^3 \; ; \; z = 0\}$ and let $P \in \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \; ; \; z = 0\}$ be a fixed point. Let K be the set which is formed by a straight line through P which moves along the curve C.

(i) Find a parametrization X with trace K.

(ii) Calculate the Gauß mapping of X. When is X a regular surface?

(*Hint: Here and for the upcoming exercises (and the upcoming sheets) you do not have to show the injectivity and the continuity of the inverse!*)

(iii) Let $P = (0, 0, 1)$. Examine the situation if C is the trace of the circle $\alpha \colon [0, 2\pi) \to \mathbb{R}^3$, $t \mapsto (\cos(t), \sin(t), 0)$ and provide a sketch.

Exercise 3.
Let

$$X \colon \mathbb{R}^2 \to \mathbb{R}^3, \; (u, v) \mapsto \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1).$$

(i) Show that X is a parameterized surface.

(ii) Calculate the Gauß mapping N of X.

(please turn the page)

(iii) Describe the vector field $V\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,1)$ along X in the form

$$V = V^1 X_u + V^2 X_v + V^3 N$$

with functions $V^k\colon \mathbb{R}^2 \to \mathbb{R}$ $(k \in \{1,2,3\})$.

(iv) Calculate the fundamental matrix G of the first fundamental form of X.

Exercise 4.
Describe the part of the unit sphere in $\mathbb{R}^3$ which is covered by the image of the Gauß mapping of the following surfaces and provide sketches:

(i) $\{(x,y,z) \in \mathbb{R}^3\ ;\ x^2 + y^2 - z = 0\}$ (Rotated paraboloid),

(ii) $\{(x,y,z) \in \mathbb{R}^3\ ;\ x^2 + y^2 - z^2 = 1\}$ (Rotated hyperboloid),

(iii) $\{(x,y,z) \in \mathbb{R}^3\ ;\ x^2 + y^2 - \cosh^2(z) = 0\}$ (Catenoide).

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.