**UNIVERSITÄT DES SAARLANDES**
**DEPARTMENT 6.1 – MATHEMATICS**
Prof. Dr. Martin Fuchs
Dr. Dominik Schillo

**Exercises for the Lecture**
**Differential Geometry**
Summer Term 2020

**Sheet 7, Solution** **Submission:** /

**Resources: Up to Lesson 13; Up to p. 56 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 143 in [Car16]**

**Exercise 1.**
Consider for $a, b, c > 0$ the subsets of $\mathbb{R}^3$:

(i) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$ (Ellipsoid),

(ii) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\right\}$ (Hyperboloid of one sheet),

(iii) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1\right\}$ (Hyperboloid of two sheets),

(iv) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0\right\}$ (Elliptic paraboloid),

(v) $\left\{(x, y, z) \in \mathbb{R}^3 \; ; \; \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0\right\}$ (Hyperbolic paraboloid).

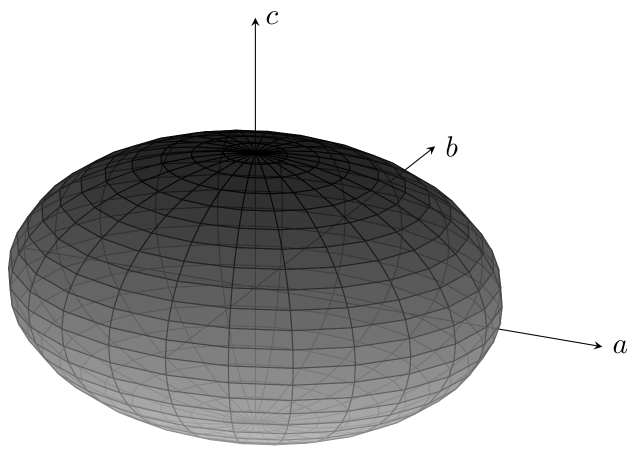
Sketch these sets and describe subsets (as large as possible) via parameterized surfaces.

**Solution 1.**

(i) A parametrization is given by

$$X\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, 2\pi) \to \mathbb{R}^3,$$
$$(\theta, \varphi) \mapsto (a\cos(\theta)\cos(\varphi), b\cos(\theta)\sin(\varphi), c\sin(\theta)).$$

Plot $(a = 2, b = 1, c = 0,5)$:

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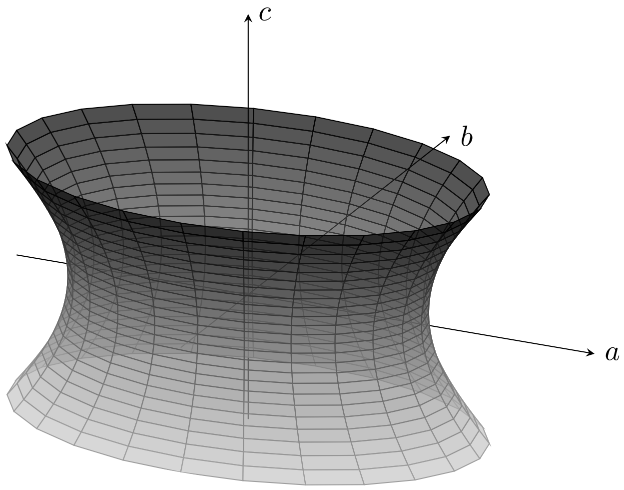
(ii) A parametrization is given by

$$X\colon \mathbb{R}\times(0,2\pi)\to\mathbb{R}^3,\ (\theta,\varphi)\mapsto(a\cosh(\theta)\cos(\varphi), b\cosh(\theta)\sin(\varphi), c\sinh(\theta)).$$

Plot ($a=b=c=1$):



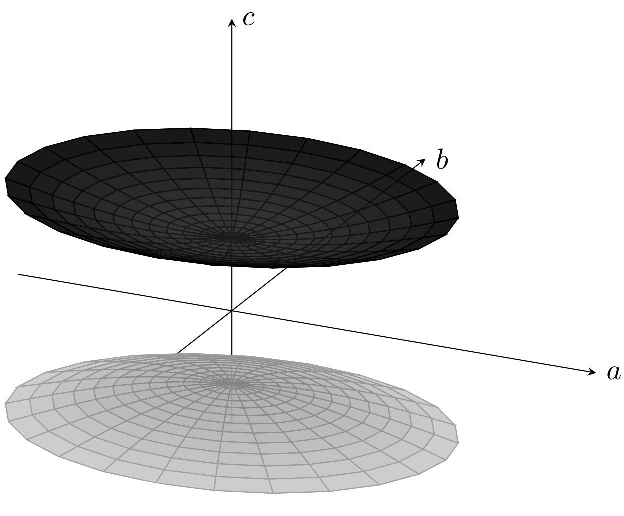

(iii) A parametrization of the upper half is given by

$$X\colon \mathbb{R}\times(0,2\pi)\to\mathbb{R}^3,\ (\theta,\varphi)\mapsto(a\sinh(\theta)\cos(\varphi), b\sinh(\theta)\sin(\varphi), c\cosh(\theta)).$$

A parametrization of the lower half is given by

$$X\colon \mathbb{R}\times(0,2\pi)\to\mathbb{R}^3,\ (\theta,\varphi)\mapsto(a\sinh(\theta)\cos(\varphi), b\sinh(\theta)\sin(\varphi), -c\cosh(\theta)).$$

Plot ($a=1, b=2, c=0,5$):




(iv) A parametrization is given by

$$X\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^3,\ (x,y)\mapsto\left(x, y, \frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Plot ($a=2, b=3$):

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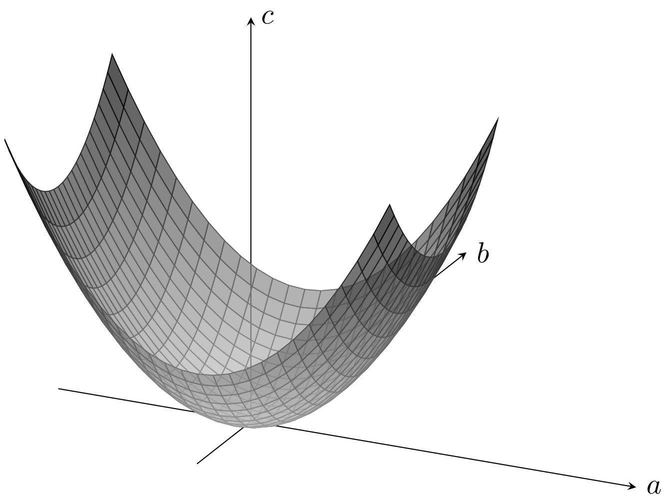
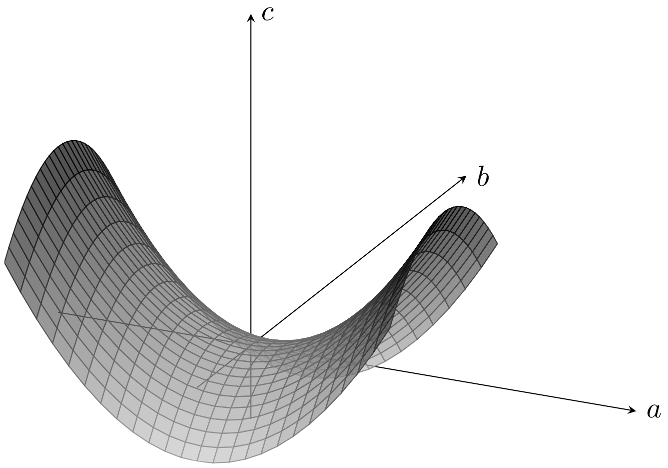



(v) A parametrization is given by

$$X\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^3,\ (x,y)\mapsto\left(x,y,\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

Plot $(a=2, b=3)$:




**Exercise 2.**
Let $I\subset\mathbb{R}$ be an interval, $C$ the trace of an regular, parameterized by arc length, injective (planar) $\alpha\colon I\to\mathbb{R}^3$ with $\mathrm{Im}(\alpha)\subset\{(x,y,z)\in\mathbb{R}^3\ ;\ z=0\}$ and let $P\in\mathbb{R}^3\setminus\{(x,y,z)\in\mathbb{R}^3\ ;\ z=0\}$ be a fixed point. Let $K$ be the set which is formed by a straight line through $P$ which moves along the curve $C$.

(i) Find a parametrization $X$ with trace $K$.

(ii) Calculate the Gauß mapping of $X$. When is $X$ a regular surface?

(*Hint: Here and for the upcoming exercises (and the upcoming sheets) you do not have to show the injectivity and the continuity of the inverse!*)

(iii) Let $P=(0,0,1)$. Examine the situation if $C$ is the trace of the circle $\alpha\colon[0,2\pi)\to\mathbb{R}^3,\ t\mapsto(\cos(t),\sin(t),0)$ and provide a sketch.

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**Solution 2.**

(i) Define

$$X\colon I^\circ \times \mathbb{R} \to \mathbb{R}^3,\ (u,v) \mapsto P + v(\alpha(u) - P).$$

(ii) Let $(u,v) \in I^\circ \times \mathbb{R}$. We have

$$X_u(u,v) = v\alpha'(u) \quad \text{und} \quad X_v(u,v) = \alpha(u) - P,$$

hence

$$(X_u \times X_v)(u,v) = v(\alpha'(u) \times (\alpha(u) - P)).$$

Therefore, $X$ is a parameterized Surface if and only if $v \neq 0$, since $\alpha'(u) \not\parallel \alpha(u) - P$ for all $u \in I^\circ$. The Gauß mapping is given by

$$N\colon I^\circ \times \mathbb{R} \setminus \{0\} \to S^2,\ (u,v) \mapsto \frac{1}{|v(\alpha'(u) \times (\alpha(u) - P))|} v(\alpha'(u) \times (\alpha(u) - P)).$$

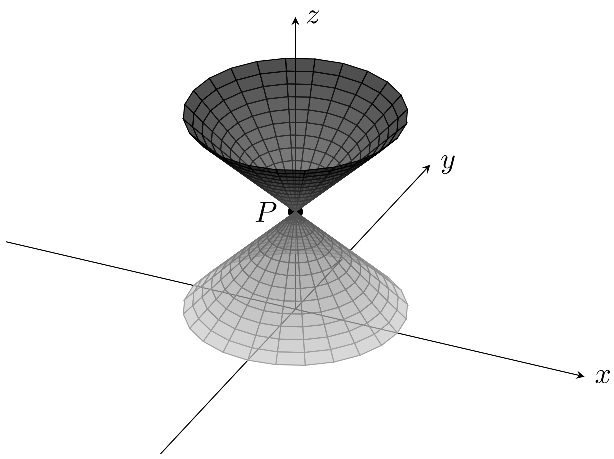
(iii) We have

$$X(u,v) = (v\cos(u), v\sin(u), 1 - v)$$

for all $(u,v) \in (0, 2\pi) \times \mathbb{R}$ and

$$N(u,v) = (-\cos(u), \sin(u), -\cos(u)^2 + \sin(u)^2)$$

for all $(u,v) \in (0,2\pi) \times (\mathbb{R} \setminus \{0\})$. We have a conical surface with apex $P$.




**Exercise 3.**

Let

$$X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1).$$

(i) Show that $X$ is a parameterized surface.

(ii) Calculate the Gauß mapping $N$ of $X$.

(iii) Describe the vector field $V\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u,v,1)$ along $X$ in the form

$$V = V^1\partial_1 X + V^2 \partial_2 X + V^3 N$$

with functions $V^k\colon \mathbb{R}^2 \to \mathbb{R}$ $(k \in \{1,2,3\})$.

(iv) Calculate the fundamental matrix $G$ of the first fundamental form of $X$.

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**Solution 3.**
Let $u, v \in \mathbb{R}$.

(i) We have

$$\partial_1 X(u,v) = \frac{-2u}{(u^2+v^2+1)^2}(2u, 2v, u^2+v^2-1) + \frac{2}{u^2+v^2+1}(1,0,u)$$
$$= \frac{2}{(u^2+v^2+1)^2}(-u^2+v^2+1, -2uv, 2u)$$

and

$$\partial_2 X(u,v) = \frac{-2v}{(u^2+v^2+1)^2}(2u, 2v, u^2+v^2-1) + \frac{2}{u^2+v^2+1}(0,1,v)$$
$$= \frac{2}{(u^2+v^2+1)^2}(-2uv, u^2-v^2+1, 2v),$$

hence

$$\langle \partial_1 X(u,v), \partial_2 X(u,v) \rangle = 0$$

and

$$|\partial_1 X(u,v)| = \frac{2}{u^2+v^2+1} = |\partial_2 X(u,v)|.$$

We obtain

$$\partial_1 X(u,v) \times \partial_2 X(u,v) = \frac{4}{(u^2+v^2+1)^3}(-2u, -2v, 1-u^2-v^2)$$

and

$$|\partial_1 X(u,v) \times \partial_2 X(u,v)| = |\partial_1 X(u,v)||\partial_2 X(u,v)| = \frac{4}{(u^2+v^2+1)^2} > 0.$$

(ii) With (i) we conclude that

$$N(u,v) = \frac{1}{|\partial_1 X(u,v) \times \partial_2 X(u,v)|}\partial_1 X(u,v) \times \partial_2 X(u,v) = \frac{1}{u^2+v^2+1}(-2u, -2v, 1-u^2-v^2).$$

(iii) Parts (i) and (ii) show that $\left(\frac{u^2+v^2+1}{2}X_u(u,v), \frac{u^2+v^2+1}{2}X_v(u,v), N(u,v)\right)$ is an orthonormal basi. Therefore,

$$V^1(u,v) = \frac{1}{|\partial_1 X(u,v)|}\left\langle V(u,v), \frac{1}{|\partial_1 X(u,v)|}\partial_1 X(u,v)\right\rangle = \frac{-u}{2}(u^2+v^2-3),$$
$$V^2(u,v) = \frac{1}{|\partial_2 X(u,v)|}\left\langle V(u,v), \frac{1}{|\partial_2 X(u,v)|}\partial_2 X(u,v)\right\rangle = \frac{-v}{2}(u^2+v^2-3),$$
$$V^3(u,v) = \langle V(u,v), N(u,v)\rangle = \frac{1-3(u^2+v^2)}{u^2+v^2+1}.$$

(iv) The fundamental matrix in $(u,v)$ is given by

$$G(u,v) = \begin{pmatrix} \frac{4}{(u^2+v^2+1)^2} & 0 \\ 0 & \frac{4}{(u^2+v^2+1)^2} \end{pmatrix} = \frac{4}{(u^2+v^2+1)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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**Exercise 4.**
Describe the part of the unit sphere in $\mathbb{R}^3$ which is covered by the image of the Gauß mapping of the following surfaces and provide sketches:

(i) $\{(x,y,z) \in \mathbb{R}^3 \; ; \; x^2 + y^2 - z = 0\}$ (Rotated paraboloid),

(ii) $\{(x,y,z) \in \mathbb{R}^3 \; ; \; x^2 + y^2 - z^2 = 1\}$ (Rotated hyperboloid),

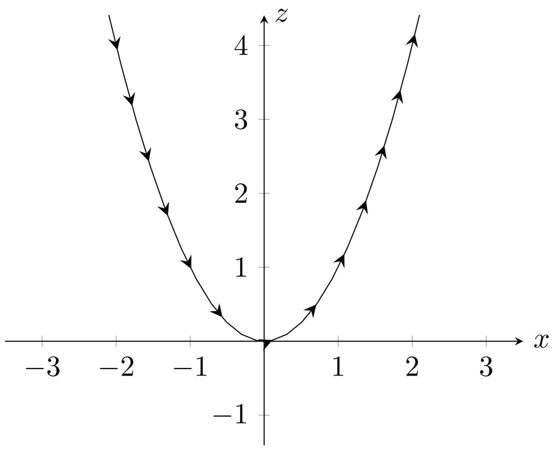
(iii) $\{(x,y,z) \in \mathbb{R}^3 \; ; \; x^2 + y^2 - \cosh^2(z) = 0\}$ (Catenoide).

**Solution 4.**
All surfaces are rotational symmetric to the $z$ axis. For $\varphi \in (0, 2\pi)$, define the rotation matrix

$$R(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Define

$$\alpha \colon \mathbb{R} \to \mathbb{R}^3, \; t \mapsto (t, 0, t^2).$$




Then

$$X \colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3, \; (u,v) \mapsto R(u)\alpha(v) = (v\cos(u), v\sin(u), v^2).$$

is a parametrization of the given set (see p. 50 in [Fuc08]) and

$$N \colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3, \; (u,v) \mapsto \frac{1}{\sqrt{1+4v^2}}(2v\cos(u), 2v\sin(u), -1)$$
$$= R(u)\frac{1}{\sqrt{1+4v^2}}(2v, 0, -1).$$

Since

$$\lim_{v\to\infty} \frac{2v}{\sqrt{1+4v^2}} = 1,$$

the image of the Gauß mapping covers the set

$$\left\{(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)) \; ; \; \theta \in \left(\frac{\pi}{2}, \pi\right], \varphi \in (0, 2\pi)\right\}.$$

Plot:

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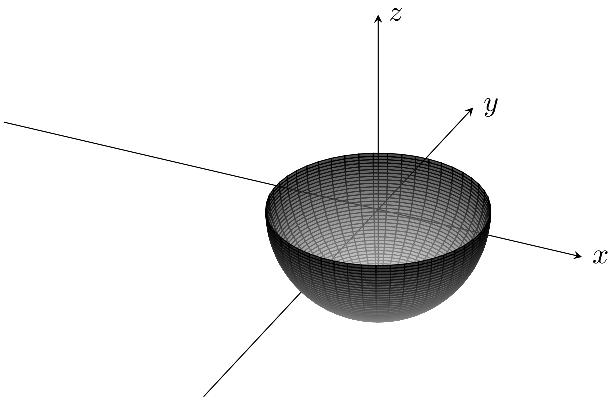
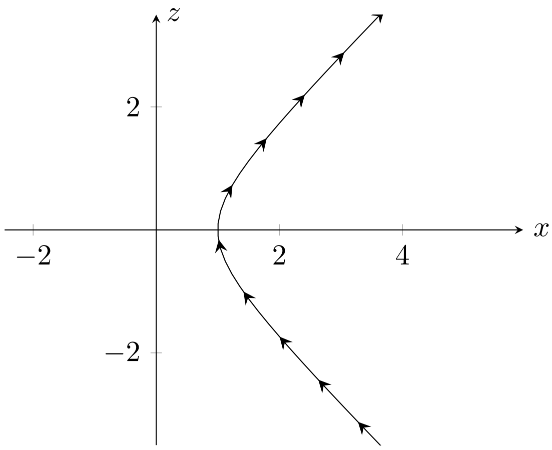



(ii) Define

$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \left(\sqrt{t^2+1}, 0, t\right).$$




Then

$$X\colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3,\ (u, v) \mapsto R(u)\alpha(v)$$

is a parametrization of the given set (see p. 50 in [Fuc08]) and

$$N\colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3,\ (u, v) \mapsto R(u)\frac{\sqrt{v^2+1}}{\sqrt{2v^2+1}}\left(1, 0, -\frac{v}{\sqrt{v^2+1}}\right).$$

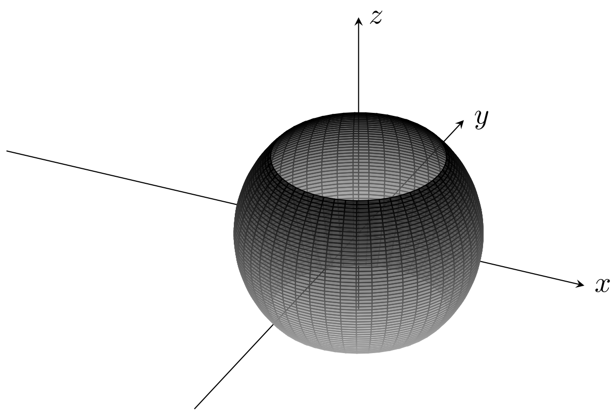
Since

$$\lim_{v\to\infty} \frac{\sqrt{v^2+1}}{\sqrt{2v^2+1}} = \lim_{v\to\infty} \frac{v}{\sqrt{2v^2+1}} = \frac{1}{\sqrt{2}},$$

the image of the Gauß mapping covers the set

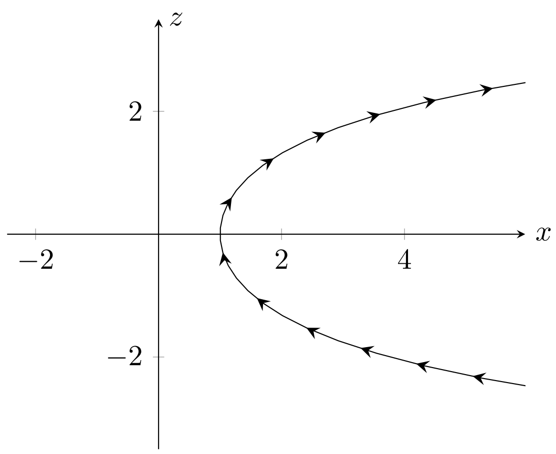
$$\left\{(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))\ ;\ \theta \in \left(\frac{\pi}{4}, \frac{3}{4}\pi\right), \varphi \in (0, 2\pi)\right\}.$$

Plot:




(iii) Define

$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto (\cosh(t), 0, t).$$




Then

$$X\colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3,\ (u, v) \mapsto R(u)\alpha(v)$$

is a parametrization of the given set (see p. 50 in [Fuc08]) and

$$N\colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3,\ (u, v) \mapsto R(u)\frac{1}{\cosh(v)}\left(1, 0, -\sinh(v)\right).$$

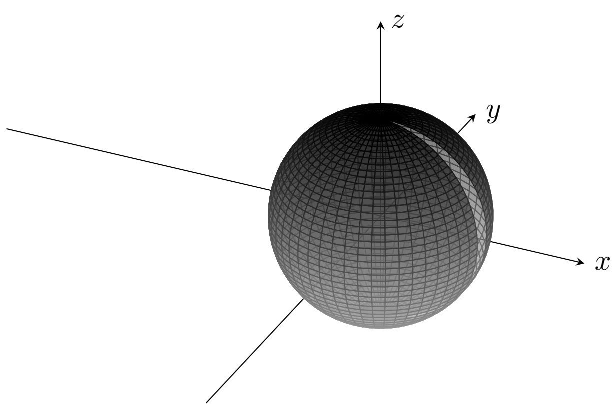
Since

$$\lim_{v\to\infty} \frac{\sinh(v)}{\cosh(v)} = 1,$$

the image of the Gauß mapping covers the set

$$\overline{B}_1(0) \setminus \{(\sin(\theta), 0, \cos(\theta))\ ;\ \theta \in [0, \pi]\}.$$

Plot (with an extra thick slit for a better visiblity):

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## References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.