UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 8 **Submission:** /

Resources: Up to Lesson 14; Up to p. 60 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 143 in [Car16]

Exercise 1.
Let $I \subset \mathbb{R}$ be an interval and let X be a surface of revolution with generating regular planar curve $\alpha\colon I \to \mathbb{R}^3,\ t \mapsto (x(t), y(t), 0)$ and with rotation around the x axis. Show that there always exists a parametrization $X\colon (0, 2\pi) \times I^\circ \to \mathbb{R}^3$ such that

$$G(u, v) = \begin{pmatrix} \mathcal{E}(v) & 0 \\ 0 & 1 \end{pmatrix}$$

for all $(u, v) \in (0, 2\pi) \times I^\circ$.

Exercise 2.
Consider the map

$$X\colon \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}^3,\ (u, v) \mapsto ((a + b\sin(v))\sin(u), (a - b\cos(v))\sin(u), c\sin(u)),$$

where a, b, c are real numbers.

(i) Determine when X is a regular parameterized surface.

(ii) Determine (in the case of regularity) the first fundamental form of X.

Exercise 3.
Let $I \subset \mathbb{R}$ be an open interval and let $\alpha\colon I \to \mathbb{R}^3$ be a regular, injective curve which is parameterized by arc length and has nowhere vanishing curvature. For $r > 0$, let

$$X\colon I \times (0, 2\pi) \to \mathbb{R}^3,\ (u, v) \mapsto \alpha(u) + r(\cos(v)n(u) + \sin(v)b(u)),$$

where n and b is the normal resp. binormal vector of the curve α.

(i) Determine the first fundamental form of X. Under which conditions is X a regular parameterized surface?

(ii) Determine the Gauß mapping of X under the assumption that X is regular.

(iii) Determine X if the curve is the circle $\alpha\colon (0, 2\pi) \to \mathbb{R}^3,\ t \mapsto (\cos(t), \sin(t), 0)$ and $r = \frac{1}{2}$. Sketch the surface.

(please turn the page)

Exercise 4.
Let $\Omega \subset \mathbb{R}^2$ be open, $X\colon \Omega \to \mathbb{R}^3$ be a parameterized surface and let $\varphi\colon \tilde{\Omega} \to \Omega$ be a parameter transformation which preserves the orientation ($\det(D\varphi) > 0$). Show the following relation between the second fundamental form II (resp. II^{TX}) of X and the second fundamental form $\widetilde{II}$ (resp. $II^{T\tilde{X}}$) of the reparameterized surface $\tilde{X} = X \circ \varphi$.

(i) For all $(\tilde{u}, \tilde{v}) \in \tilde{\Omega}$ and $\tilde{U}, \tilde{V} \in \mathbb{R}^2$, we have

$$\widetilde{II}_{(\tilde{u},\tilde{v})}(\tilde{U}, \tilde{V}) = II_{\varphi(\tilde{u},\tilde{v})}(D\varphi_{(\tilde{u},\tilde{v})}\tilde{U}, D\varphi_{(\tilde{u},\tilde{v})}\tilde{V}).$$

(ii) For all $(\tilde{u}, \tilde{v}) \in \tilde{\Omega}$ and $U, V \in T_{(\tilde{u},\tilde{v})}\tilde{X}$, we have

$$II^{T\tilde{X}}_{(\tilde{u},\tilde{v})}(U, V) = II^{TX}_{\varphi(\tilde{u},\tilde{v})}(U, V).$$

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.