UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 8, Solution **Submission:** /

Resources: Up to Lesson 14; Up to p. 60 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 143 in [Car16]

Exercise 1.
Let $I \subset \mathbb{R}$ be an interval and let X be a surface of revolution with generating regular planar curve $\alpha\colon I \to \mathbb{R}^3$, $t \mapsto (x(t), y(t), 0)$ and with rotation around the x axis. Show that there always exists a parametrization $X\colon (0, 2\pi) \times I^\circ \to \mathbb{R}^3$ such that

$$G(u,v) = \begin{pmatrix} \mathcal{E}(v) & 0 \\ 0 & 1 \end{pmatrix}$$

for all $(u, v) \in (0, 2\pi) \times I^\circ$.

Solution 1.
Since the surface of revolution is independent of the parametrization of the curve, without loss of generality, we can assume that α is parameterized by arc length. The surface of revolution can be parameterized by

$$X\colon (0, 2\pi) \times I^\circ \to \mathbb{R}^3,\ (u,v) \mapsto (x(v), \cos(u)y(v), \sin(u)y(v)).$$

Since

$$\partial_1 X(u,v) = (0, -\sin(u)y(v), \cos(u)y(v)) \quad \text{and} \quad \partial_2 X(u,v) = (x'(v), \cos(u)y'(v), \sin(u)y'(v))$$

for all $(u, v) \in (0, 2\pi) \times I^\circ$, it follows that

$$G(u,v) = \begin{pmatrix} y(v)^2 & 0 \\ 0 & (x'(v))^2 + (y'(v))^2 \end{pmatrix} = \begin{pmatrix} y(v)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $(u, v) \in (0, 2\pi) \times I^\circ$.

Exercise 2.
Consider the map

$$X\colon \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}^3,\ (u,v) \mapsto ((a + b\sin(v))\sin(u), (a - b\cos(v))\sin(u), c\sin(u)),$$

where a, b, c are real numbers.

(i) Determine when X is a regular parameterized surface.

(ii) Determine (in the case of regularity) the first fundamental form of X.

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Solution 2.
We have

$$\partial_1 X(u,v) = \cos(u)(a + b\sin(v), a - b\cos(v), c) \quad \text{and} \quad \partial_2 X(u,v) = b\sin(u)(\cos(v), \sin(v), 0)$$

for all $(u,v) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$. Hence we obtain

$$\begin{aligned}
|\partial_1 X(u,v)|^2 &= \cos(u)^2 \left(\left(\sqrt{2}a + b\sin\left(v - \frac{\pi}{4}\right)\right)^2 + b^2 \sin\left(v + \frac{\pi}{4}\right)^2 + c^2 \right), \\
|\partial_2 X(u,v)|^2 &= b^2 \sin(u)^2, \\
\langle \partial_1 X(u,v), \partial_2 X(u,v) \rangle &= ab\sin(u)\cos(u)(\cos(v) + \sin(v)), \\
\partial_1 X(u,v) \times \partial_2 X(u,v) &= b\sin(u)\cos(u) \begin{pmatrix} -c\sin(v) \\ c\cos(v) \\ a(\sin(v) - \cos(v)) + b\sin(v)(\sin(v) + \cos(v)) \end{pmatrix}
\end{aligned}$$

for all $(u,v) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$. Since $\sin(t) \neq 0 \neq \cos(t)$ and $\sin(v) + \cos(v) > 0$ for all $t \in \left(0, \frac{\pi}{2}\right)$, X is (locally) regular if $a, b, c \in \mathbb{R}$ with $b \neq 0$. Furthermore, we have

$$G(u,v) = \begin{pmatrix} \cos(u)^2 \left(\left(\sqrt{2}a + b\sin\left(v - \frac{\pi}{4}\right)\right)^2 + b^2 \sin\left(v + \frac{\pi}{4}\right)^2 + c^2 \right) & ab\sin(u)\cos(u)(\cos(v) + \sin(v)) \\ ab\sin(u)\cos(u)(\cos(v) + \sin(v)) & b^2\sin(u)^2 \end{pmatrix}.$$

Exercise 3.
Let $I \subset \mathbb{R}$ be an open interval and let $\alpha\colon I \to \mathbb{R}^3$ be a regular, injective curve which is parameterized by arc length and has nowhere vanishing curvature. For $r > 0$, let

$$X\colon I \times (0, 2\pi) \to \mathbb{R}^3, \ (u,v) \mapsto \alpha(u) + r(\cos(v)n(u) + \sin(v)b(u)),$$

where n and b is the normal resp. binormal vector of the *directrix* α.

(i) Determine the first fundamental form of X. Under which conditions is X a regular parameterized surface?

(ii) Determine the Gauß mapping of X under the assumption that X is regular.

(iii) Determine X if the directrix is the circle $\alpha\colon (0, 2\pi) \to \mathbb{R}^3, \ t \mapsto (\cos(t), \sin(t), 0)$ and $r = \frac{1}{2}$. Sketch the surface.

Solution 3.
Let $(u,v) \in I \times (0, 2\pi)$.

(i) We have

$$\partial_1 X(u,v) = (1 - r\cos(v)\kappa(u))t(u) + r\sin(v)\tau(u)n(u) - r\cos(v)\tau(u)b(u)$$

and

$$\partial_2 X(u,v) = -r\sin(v)n(u) + r\cos(v)b(u)$$

hence

$$|\partial_1 X(u,v)|^2 = (1 - r\cos(v)\kappa(u))^2 + r^2\tau(u)^2$$

and

$$|\partial_2 X(u,v)|^2 = r^2$$

as well as

$$\langle \partial_1 X(u,v), \partial_2 X(u,v) \rangle = -r^2\tau(u).$$

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Therefore it follows that

$$\partial_1 X(u,v) \times \partial_2 X(u,v) = -r\cos(v)(1 - r\cos(v)\kappa(u))n(u) - r\sin(v)(1 - r\cos(v)\kappa(u))b(u)$$

and

$$|\partial_1 X(u,v) \times \partial_2 X(u,v)|^2 = r^2(1 - r\cos(v)\kappa(u))^2.$$

Hence we obtain

$$\partial_1 X(u,v) \times \partial_2 X(u,v) = 0 \iff \cos(v) = \frac{1}{r\kappa(u)}.$$

Thus X is regular if $\kappa \le \frac{1}{r}$, since $\cos(v) < 1$.

(ii) With (i) we obtain

$$N\colon I \times (0, 2\pi) \to S^2,\ (u,v) \mapsto -\cos(v)n(u) - \sin(v)b(u).$$

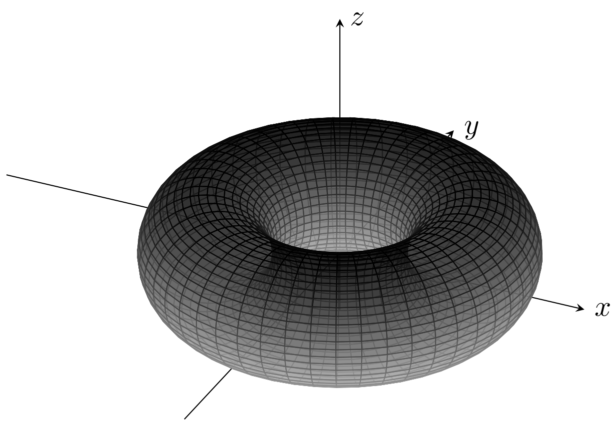
(iii) We have

$$\begin{aligned} t(u) &= (-\sin(u), \cos(u), 0), \\ n(u) &= (-\cos(u), -\sin(u), 0) = -\alpha(u), \\ b(u) &= t(u) \times n(u) = (0,0,1), \\ \kappa(u) &= 1, \\ \tau(u) &= 0. \end{aligned}$$

It follows that

$$X(u,v) = \left(\cos(u)\left(1 - \frac{1}{2}\cos(v)\right), \sin(u)\left(1 - \frac{1}{2}\cos(v)\right), \frac{1}{2}\sin(v)\right).$$

Plot:



Exercise 4.
Let $\Omega \subset \mathbb{R}^2$ be open, $X\colon \Omega \to \mathbb{R}^3$ be a parameterized surface and let $\varphi\colon \tilde{\Omega} \to \Omega$ be a parameter transformation which preserves the orientation ($\det(D\varphi) > 0$). Show the following relation between the second fundamental form II (resp. II^{TX}) of X and the second fundamental form $\widetilde{II}$ (resp. $II^{T\tilde{X}}$) of the reparameterized surface $\tilde{X} = X \circ \varphi$.

(i) For all $(\tilde{u}, \tilde{v}) \in \tilde{\Omega}$ and $\tilde{U}, \tilde{V} \in \mathbb{R}^2$, we have

$$\widetilde{II}_{(\tilde{u},\tilde{v})}(\tilde{U}, \tilde{V}) = II_{\varphi(\tilde{u},\tilde{v})}(D\varphi_{(\tilde{u},\tilde{v})}\tilde{U}, D\varphi_{(\tilde{u},\tilde{v})}\tilde{V}).$$

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(ii) For all $(\tilde{u},\tilde{v}) \in \tilde{\Omega}$ and $U, V \in T_{(\tilde{u},\tilde{v})}\tilde{X}$, we have

$$II^{T\tilde{X}}_{(\tilde{u},\tilde{v})}(U,V) = II^{TX}_{\varphi(\tilde{u},\tilde{v})}(U,V).$$

Solution 4.

(i) Let $(\tilde{u},\tilde{v}) \in \tilde{\Omega}$ and $\tilde{U}, \tilde{V} \in \mathbb{R}^2$. Since φ preserves orientation, we have $\tilde{N} = N \circ \varphi$ and therfore

$$\begin{aligned}
\widetilde{II}_{(\tilde{u},\tilde{v})}(\tilde{U},\tilde{V}) &= -D\tilde{N}_{(\tilde{u},\tilde{v})}(\tilde{U})D\tilde{X}_{(\tilde{u},\tilde{v})}(\tilde{V}) \\
&= -D(N\circ\varphi)_{(\tilde{u},\tilde{v})}(\tilde{U})D(X\circ\varphi)_{(\tilde{u},\tilde{v})}(\tilde{V}) \\
&= -DN_{\varphi(\tilde{u},\tilde{v})}D\varphi_{(\tilde{u},\tilde{v})}(\tilde{U})DX_{\varphi(\tilde{u},\tilde{v})}D\varphi_{(\tilde{u},\tilde{v})}(\tilde{V}) \\
&= II_{\varphi(\tilde{u},\tilde{v})}(D\varphi(\tilde{u},\tilde{v})\tilde{U}, D\varphi(\tilde{u},\tilde{v})\tilde{V}).
\end{aligned}$$

(ii) Let $(\tilde{u},\tilde{v}) \in \tilde{\Omega}$ and $U, V \in T_{(\tilde{u},\tilde{v})}\tilde{X}$. Since φ preserves orientation, we have $\tilde{N} = N \circ \varphi$ and therefore

$$\begin{aligned}
II^{T\tilde{X}}_{(\tilde{u},\tilde{v})}(U,V) &= -D\tilde{N}_{(\tilde{u},\tilde{v})}((D\tilde{X}_{(\tilde{u},\tilde{v})})^{-1}U)V \\
&= -D(N\circ\varphi)_{(\tilde{u},\tilde{v})}((D(X\circ\varphi)_{(\tilde{u},\tilde{v})})^{-1}U)V \\
&= -DN_{\varphi(\tilde{u},\tilde{v})}D\varphi_{(\tilde{u},\tilde{v})}((DX_{\varphi(\tilde{u},\tilde{v})}D\varphi_{(\tilde{u},\tilde{v})})^{-1}U)V \\
&= -DN_{\varphi(\tilde{u},\tilde{v})}D\varphi_{(\tilde{u},\tilde{v})}(D\varphi_{(\tilde{u},\tilde{v})})^{-1}((DX_{\varphi(\tilde{u},\tilde{v})})^{-1}U)V \\
&= -DN_{\varphi(\tilde{u},\tilde{v})}((DX_{\varphi(\tilde{u},\tilde{v})})^{-1}U)V \\
&= II^{TX}_{\varphi(\tilde{u},\tilde{v})}(U,V).
\end{aligned}$$

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces.* Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie.* 2008.