UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
Prof. Dr. Martin Fuchs
Dr. Dominik Schillo

Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 9 **Submission:** /

Resources: Up to Lesson 15; Up to p. 65 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 147 in [Car16]

Exercise 1.
Consider the parametrization

$$X: (-\pi, \pi) \times \mathbb{R},\ (u, v) \mapsto (\cos(u), \sin(u), v),$$

i.e. X is a parametrization of the cylinder

$$Z = \{(x, y, z) \in \mathbb{R}^3\ ;\ x^2 + y^2 = 1\}.$$

Determine all normal sections as well as the minimal and the maximal normal curvature of X in $p = X(0, 0) = (1, 0, 0)$.

Exercise 2.
(*See Exercise 18 in Section 3-2 in [Car16]*)

Show: If a regular curve C is the intersection of two surfaces X_1 X_2, then the curvature $\kappa_C(p)$ of C in $p \in C$ is given by

$$\kappa_C(p)^2 \sin(\theta)^2 = \kappa_{n_1}^2 + \kappa_{n_2}^2 - 2\kappa_{n_1}\kappa_{n_2}\cos(\theta),$$

where κ_{n_1} and κ_{n_2} are the normal curvatures at p along the tangent on C of X_1 resp. X_2 and θ is the angle between the normal vectors N_1 and N_2 of X_1 resp. X_2 at p.

(*Hint: Consider the triangle spanned by* $\kappa_{n_1}N_2$ *and* $\kappa_{n_2}N_1$*.*)

Exercise 3.
Let a and r be positive real numbers with $r < a$ and define the torus

$$X: (0, 2\pi) \times (-\pi, \pi) \to \mathbb{R}^3,\ (u, v) \mapsto ((a + r\cos(u))\cos(v), (a + r\cos(u))\sin(v), r\sin(u)).$$

(i) Calculate the first and second fundamental form of X.

(ii) Determine a formula for the area of X.

(iii) Sketch the normal sections of X through the point $(a, 0, r)$.

Exercise 4.
Let I be an open interval, $\alpha: I \to \mathbb{R}^3$ be a regular curve and let $w: I \to \mathbb{R}^3 \setminus \{0\}$ be smooth. The mapping

$$X: I \times \mathbb{R} \to \mathbb{R}^3,\ (u, v) \mapsto \alpha(u) + vw(u)$$

is called a *ruled surface* if X is regular. The curve α is called *directrix* and the straight lines $\mathbb{R} \to \mathbb{R}^3,\ v \mapsto \alpha(u) + vw(u)$ $(u \in I)$ are called *generators*.

(please turn the page)

(i) Under which conditions is X a regular parameterized surface?

(ii) Let $u \in I$ such that the tangent plane $T_{(u,v)}X$ of X in (u, v) exists for all $v \in \mathbb{R}$. Show that $T_{(u,v)}X = T_{(u,\tilde{v})}X$ for all $v, \tilde{v} \in \mathbb{R}$ if and only if the vectors $\alpha'(u)$, $w'(u)$ and $w(u)$ are linear dependent.

(iii) Show that hyperbolic paraboloid (see Sheet 7, Exercise 1) is a ruled surface.

(*Hint: Third binomial formula.*)

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.