UNIVERSITÄT DES SAARLANDES
DEPARTMENT 6.1 – MATHEMATICS
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Exercises for the Lecture
Differential Geometry
Summer Term 2020

Sheet 9, Solution **Submission:** /

Resources: Up to Lesson 15; Up to p. 65 in [Fuc08]; Sections 2-1 – 2-5 and Section 3-1 – p. 147 in [Car16]

Exercise 1.
Consider the parametrization

$$X\colon (-\pi,\pi)\times\mathbb{R},\ (u,v)\mapsto(\cos(u),\sin(u),v),$$

i.e. X is a parametrization of the cylinder

$$Z=\{(x,y,z)\in\mathbb{R}^3\ ;\ x^2+y^2=1\}.$$

Determine all normal sections as well as the minimal and the maximal normal curvature of X in $p=X(0,0)=(1,0,0)$.

Solution 1.
We have

$$\begin{aligned}\partial_1X(u,v)&=(-\sin(u),\cos(u),0),\\ \partial_2X(u,v)&=(0,0,1)\end{aligned}$$

and

$$\partial_1X(u,v)\times\partial_2X(u,v)=(\cos(u),\sin(u),0)$$

for all $(u,v)\in(-\pi,\pi)\times\mathbb{R}$. Let $\theta\in[0,\frac{\pi}{2}]$. We consider the plane

$$E_\theta=\{p+\lambda(\partial_1X(0,0)\times\partial_2X(0,0))+\mu(\cos(\theta)\partial_1X(0,0)+\sin(\theta)\partial_2X(0,0))\ ;\ \lambda,\mu\in\mathbb{R}\}.$$

Then

$$\begin{aligned}n_\theta&=(\partial_1X(0,0)\times\partial_2X(0,0))\times(\cos(\theta)\partial_1X(0,0)+\sin(\theta)\partial_2X(0,0))\\ &=\begin{pmatrix}1\\0\\0\end{pmatrix}\times\left(\cos(\theta)\begin{pmatrix}0\\1\\0\end{pmatrix}+\sin(\theta)\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\\ &=\begin{pmatrix}0\\-\sin(\theta)\\\cos(\theta)\end{pmatrix}\end{aligned}$$

is the normal vector of this plane and, since

$$0=\langle n_\theta,X(u,v)-p\rangle=-\sin(\theta)\sin(u)+\cos(\theta)v\quad((u,v)\in(-\pi,\pi)\times\mathbb{R}),$$

we obtain for the normal section

$$\mathrm{Bild}(X)\cap E_\theta=\{(u,v)\in(-\pi,\pi)\times\mathbb{R}\ ;\ \cos(\theta)v=\sin(\theta)\sin(u)\}.$$

Let $(u,v)\in\mathrm{Bild}(X)\cap E_\theta$.

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(i) $\theta \in [0, \frac{\pi}{2})$: Then
$$v = \tan(\theta)\sin(u),$$
hence
$$\alpha\colon (-\pi, \pi) \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \\ \tan(\theta)\sin(t) \end{pmatrix}$$
is a parametrization of the normal section. For $t \in (-\pi, \pi)$, we obtain
$$\alpha'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ \tan(\theta)\cos(t) \end{pmatrix},\ \alpha''(t) = \begin{pmatrix} -\cos(t) \\ -\sin(t) \\ -\tan(\theta)\sin(t) \end{pmatrix},\ \alpha'(t) \times \alpha''(t) = \begin{pmatrix} 0 \\ -\tan(\theta) \\ 1 \end{pmatrix},$$
thus
$$\kappa_\alpha(t) = \frac{(1 + \tan(\theta)^2)^{1/2}}{(1 + \tan(\theta)^2\cos(t)^2)^{3/2}}$$
and therefore
$$\kappa_\alpha(0) = \frac{1}{1 + \tan(\theta)^2} \in (0, 1].$$

(ii) $\theta = \frac{\pi}{2}$: Then
$$0 = -\sin(\theta)\sin(u) + \cos(\theta)v = -\sin(u) \iff u = 0,$$
hence
$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}$$
is a parametrization of the normal section. For $t \in \mathbb{R}$, we obtain
$$\alpha'(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \alpha''(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ \alpha'(t) \times \alpha''(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
thus
$$\kappa_\alpha(t) = 0 = \kappa_\alpha(0).$$

The minimal and maximal normal curvature of X in p is therefore 0 and 1, respectively.

Exercise 2.

(*See Exercise 18 in Section 3-2 in [Car16]*)

Show: If a regular curve C is the intersection of two surfaces X_1 X_2, then the curvature $\kappa_C(p)$ of C in $p \in C$ is given by
$$\kappa_C(p)^2 \sin(\theta)^2 = \kappa_{n_1}^2 + \kappa_{n_2}^2 - 2\kappa_{n_1}\kappa_{n_2}\cos(\theta),$$
where κ_{n_1} and κ_{n_2} are the normal curvatures at p along the tangent on C of X_1 resp. X_2 and θ is the angle between the normal vectors N_1 and N_2 of X_1 resp. X_2 at p.

(*Hint: Consider the triangle spanned by $\kappa_{n_1}N_2$ and $\kappa_{n_2}N_1$.*)

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Solution 2.
We consider the triangle spanned by $\kappa_{n_1} N_2$ and $\kappa_{n_2} N_1$. For the third side, we obtain by the law of cosines

$$|\kappa_{n_1} N_2 - \kappa_{n_2} N_1|^2 = \kappa_{n_1}^2 + \kappa_{n_2}^2 - 2\cos(\theta)\kappa_{n_1}\kappa_{n_2}.$$

With Sheet 2, Exercise 1 (iii) we obtain

$$\begin{aligned} |\kappa_{n_1} N_2 - \kappa_{n_2} N_1| &= |\kappa_C(p)\langle n, N_1\rangle N_2 - \kappa_C(p)\langle n, N_2\rangle N_1| \\ &= \kappa_C(p)|\langle n, N_1\rangle N_2 - \langle n, N_2\rangle N_1| \\ &= \kappa_C(p)|n \times (N_1 \times N_2)| \\ &= \kappa_C(p)|N_1 \times N_2| \\ &= \kappa_C(p)|\sin(\theta)|, \end{aligned}$$

where we used $n \perp N_1 \times N_2 = t_C(p)$ and $|n| = |N_1| = |N_2|$. Hence the result follows.

Exercise 3.
Let a and r be positive real numbers with $r < a$ and define the torus

$$X\colon (0, 2\pi) \times (-\pi, \pi) \to \mathbb{R}^3,\ (u, v) \mapsto ((a + r\cos(u))\cos(v), (a + r\cos(u))\sin(v), r\sin(u)).$$

(i) Calculate the first and second fundamental form of X.

(ii) Determine a formula for the area of X.

(iii) Sketch the normal sections of X through the point $(a, 0, r)$.

Solution 3.
Let $(u, v) \in (0, 2\pi) \times (-\pi, \pi)$.

(i) We have

$$\begin{aligned} \partial_1 X(u, v) &= -r(\sin(u)\cos(v), \sin(u)\sin(v), -\cos(u)), \\ \partial_2 X(u, v) &= (a + r\cos(u))(-\sin(v), \cos(v), 0), \end{aligned}$$

hence

$$\begin{aligned} |\partial_1 X(u, v)|^2 &= r^2, \\ |\partial_2 X(u, v)|^2 &= (a + r\cos(u))^2, \\ \langle \partial_1 X(u, v), \partial_2 X(u, v)\rangle &= 0. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \partial_1 X(u, v) \times \partial_2 X(u, v) &= -r(a + r\cos(u))(\cos(u)\cos(v), \cos(u)\sin(v), \sin(u)), \\ |\partial_1 X(u, v) \times \partial_2 X(u, v)| &= r(a + r\cos(u)), \end{aligned}$$

and therfore

$$N(u, v) = -(\cos(u)\cos(v), \cos(u)\sin(v), \sin(u)).$$

Furthermore, we see that

$$\begin{aligned} \partial_1 N(u, v) &= (\cos(v)\sin(u), \sin(v)\sin(u), -\cos(u)), \\ \partial_2 N(u, v) &= (\sin(v)\cos(u), -\cos(v)\cos(u), 0). \end{aligned}$$

The first fundamental form is given by

$$\begin{pmatrix} r^2 & 0 \\ 0 & (a + r\cos(u))^2 \end{pmatrix}$$

and the second fundamental form is given by

$$\begin{pmatrix} r & 0 \\ 0 & (a + r\cos(u))\cos(u) \end{pmatrix}.$$

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(ii) With (i) we obtain

$$\begin{aligned} A_{(0,2\pi)\times(-\pi,\pi)}(X) &= \int_{(0,2\pi)\times(-\pi,\pi)} |\partial_1 X(u,v) \times \partial_2 X(u,v)| \, \mathrm{d}(u,v) \\ &= \int_{(-\pi,\pi)} \int_{(0,2\pi)} r(a + r\cos(u)) \, \mathrm{d}u\mathrm{d}v \\ &= 2\pi r[au + r\sin(u)]_0^{2\pi} \\ &= 4\pi^2 ra. \end{aligned}$$

(iii) Cassini ovals.

Exercise 4.
Let I be an open interval, $\alpha\colon I \to \mathbb{R}^3$ be a regular curve and let $w\colon I \to \mathbb{R}^3 \setminus \{0\}$ be smooth. The mapping

$$X\colon I \times \mathbb{R} \to \mathbb{R}^3, \ (u,v) \mapsto \alpha(u) + vw(u)$$

is called a *ruled surface* if X is regular. The curve α is called *directrix* and the straight lines $\mathbb{R} \to \mathbb{R}^3, \ v \mapsto \alpha(u) + vw(u)$ $(u \in I)$ are called *generators*.

(i) Under which conditions is X a regular parameterized surface?

(ii) Let $u \in I$ such that the tangent plane $T_{(u,v)}X$ of X in (u,v) exists for all $v \in \mathbb{R}$. Show that $T_{(u,v)}X = T_{(u,\tilde{v})}X$ for all $v, \tilde{v} \in \mathbb{R}$ if and only if the vectors $\alpha'(u)$, $w'(u)$ and $w(u)$ are linear dependent.

(iii) Show that hyperbolic paraboloid (see Sheet 7, Exercise 1) is a ruled surface.

(*Hint: Third binomial formula.*)

Solution 4.

(i) Let $(u,v) \in I \times \mathbb{R}$. We have

$$\begin{aligned} \partial_1 X(u,v) &= \alpha'(u) + vw'(u), \\ \partial_2 X(u,v) &= w(u), \end{aligned}$$

hence

$$\partial_1 X(u,v) \times \partial_2 X(u,v) = \alpha'(u) \times w(u) + vw'(u) \times w(u).$$

Therfore X is regular if and only if $\alpha'(u) \times w(u)$ and $w'(u) \times w(u)$ are linear independent for all $u \in I$.

(ii) First, let $T_{(u,v)}X = T_{(u,\tilde{v})}X$ for all $v, \tilde{v} \in \mathbb{R}$. It follows that

$$T_{(u,v)}X = T_{(u,0)}X = \operatorname{span}\{\alpha'(u), w(u)\}$$

for all $v \in \mathbb{R}$ and hence

$$\partial_1 X(u,v) = \alpha'(u) + vw'(u) = \lambda\alpha'(u) + \mu w(u)$$

for all $v \in \mathbb{R}$. We obtain that

$$0 = (\lambda - 1)\alpha'(u) + \mu w(u) - vw'(u)$$

for all $v \in \mathbb{R}$. Thus $\alpha'(u), w(u), w'(u)$ are linear dependent.

Now let $\alpha'(u), w(u), w'(u)$ be linear dependent. Then $\alpha'(u) \times w(u)$ and $w'(u) \times w(u)$ are linear dependent and thus $\partial_1 X(u,v) \times \partial_2 X(u,v)$ and $\alpha'(u) \times w(u)$ are linear dependent for all $v \in \mathbb{R}$. We conclude that

$$T_{(u,v)}X = \{\partial_1 X(u,v) \times \partial_2 X(u,v)\}^\perp = \{\alpha'(u) \times w(u)\}^\perp = T_{(u,\tilde{v})}X$$

for all $v, \tilde{v} \in \mathbb{R}$.

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(iii) The hyperbolic paraboloid has been parameterized by

$$X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(u, v, \frac{u^2}{a^2} - \frac{v^2}{b^2}\right).$$

Since

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = \left(\frac{u}{a} - \frac{v}{b}\right)\left(\frac{u}{a} + \frac{v}{b}\right)$$

for all $(u,v) \in \mathbb{R}^2$, using the linear bijection

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (u,v) \mapsto \left(\frac{u}{a} - \frac{v}{b}, \frac{u}{a} + \frac{v}{b}\right) = \begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \\ \frac{1}{a} & \frac{1}{b} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

we can write the parametrization as

$$\begin{aligned} X\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) &\mapsto (\varphi^{-1}(\varphi(u,v)), \varphi_1(u,v)\varphi_2(u,v)) \\ &= \left(\frac{a}{2}(\varphi_1(u,v) + \varphi_2(u,v)), \frac{b}{2}(\varphi_2(u,v) - \varphi_1(u,v)), \varphi_1(u,v)\varphi_2(u,v)\right) \end{aligned}$$

or

$$\tilde{X}\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (\tilde{u},\tilde{v}) \mapsto \left(\frac{a}{2}(\tilde{u}+\tilde{v}), \frac{b}{2}(\tilde{v}-\tilde{u}), \tilde{u}\tilde{v}\right) = \left(\frac{a}{2}\tilde{u}, -\frac{b}{2}\tilde{u}, 0\right) + \tilde{v}\left(\frac{a}{2}, \frac{b}{2}, \tilde{u}\right),$$

since

$$\varphi^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (\tilde{u},\tilde{v}) \mapsto \begin{pmatrix} \frac{a}{2} & \frac{a}{2} \\ -\frac{b}{2} & \frac{b}{2} \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \left(\frac{a}{2}(\tilde{u}+\tilde{v}), \frac{b}{2}(\tilde{v}-\tilde{u})\right).$$

With

$$\alpha\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \left(\frac{a}{2}t, -\frac{b}{2}t, 0\right) \quad \text{and} \quad w\colon \mathbb{R} \to \mathbb{R}^3 \setminus \{0\},\ t \mapsto \left(\frac{a}{2}, \frac{b}{2}, t\right)$$

it follows that

$$\tilde{X}(\tilde{u},\tilde{v}) = \alpha(\tilde{u}) + vw(\tilde{u})$$

for all $(\tilde{u},\tilde{v}) \in \mathbb{R}^2$.

References

[Car16] Manfredo P. do Carmo. *Differential geometry of curves & surfaces*. Revised & updated second edition. Dover Publications, Inc., Mineola, NY, 2016.

[Fuc08] Martin Fuchs. *Vorlesungsskript zur Differentialgeometrie*. 2008.