Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
Dr. Darya Apushkinskaya

Calculus of Variations (Summer Term 2014)
Exam-Part W

Problem E.1 (10 Points)

Find the extremals of the functional

$$J[y] = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y} dx$$

which connect the origin to the vertical line $x = 1$. Describe (simply) the shape of this curve.

Problem E.2 (10 Points)

Solve the isoperimetric problem inside a square region, i.e., what is the shape that contains the largest area without exceeding a given perimeter L, given that the shape must be entirely contained in a square with sides $2W$ in length.

Note that the problem is uninteresting for $W > \dfrac{L}{2\pi}$ because a circle of radius $R = \dfrac{L}{2\pi}$ satisfies the isoperimetric constraint, and fits inside the square, and this is clearly the maximal area region (though there are actually multiple possible circles that might fit).

Likewise, $8W < L$ is uninteresting, because we cannot meet the perimeter constraint without having a concave shape, so the obvious solution is to contain the entire area of the square, but have the perimeter dip into the shape along a line enclosing zero area.

So, we consider the case

$$\frac{L}{8} < W < \frac{L}{2\pi}.$$

(*Hint:* Think about symmetry of the problem)

Problem E.3 (10 Points)

Consider the functional

$$J[y] = \int_0^2 \left[(y'^2 + y^2)(y - x)^2 - \frac{4}{3}y^3 + 2xy^2 \right] dx$$

$$y(0) = 0, \qquad y(2) = e$$

a) Find the corresponding Euler-Lagrange equation and find out at what points in $[0, 2]$ it must hold.

b) Show that $y = x$ and $y = \alpha e^x$ are solutions of the equation found in part a).

c) Find all possible corner points for the extremals of J.

d) Find a possible extremal for our problem.

Problem E.4 (10 Points)

Use Ritz's method to find an approximate solution to minimize the

$$J[y] = \int_0^{2\pi} \left[y'^2 + \lambda^2 y^2 \right] dx,$$

where $y(0) = 1$ and $y(2\pi) = 1$ and λ is a positive integer. Use the trial functions

$$\phi_n(x) = \cos(nx).$$

Compare your solution to one found directly from the Euler-Lagrange equations.

Problem E.5 (10 Points)

Solve the following optimal control problem: find the control $0 \leqslant u(t) \leqslant 1$ that minimizes

$$J[u] = \int_0^T (x_1 u - x_2 u)\, dt$$

subject to the system DEs

$$\dot{x}_1 = 1 - u$$
$$\dot{x}_2 = x_1 + 1$$

Given starting point $(x_1, x_2) = (0, 0)$, and end-point $(x_1, x_2) = (1, 2)$ derive the time T at which we reach the end-point.

Deadline for submission: Monday, October 13, 2014