Calculus of Variations
Summer Term 2014

Lecture 15

27. Juni 2014

Purpose of Lesson:

- To consider optimal control examples
- To introduce a terminology.

Formulation of control problems

We break a control problems into two parts

1. The system state: $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^t$

   The system state describes the system (e.g. position and velocity of the car in car parking example)

2. The control: $\mathbf{u}(t) = (u_1(t), \dots, u_m(t))^t$

   We apply the control to the system (e.g. force applied to the car).

The evolution of the system is governed by the set of DEs

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

In a control problem we want to get the system to a particular state $\mathbf{x}(t)$ at time t, given initial state $\mathbf{x}(t_0)$.

Optimal control problems

- In an optimal control problem we have still have the system equations
$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$
and we might wish to get to state $\mathbf{x}(t)$ given initial state $\mathbf{x}(t_0)$, but now we wish to do so while minimizing a functional
$$J[\mathbf{x}, \mathbf{u}] = \int\limits_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u}) dt.$$
- That is, we wish to choose a function $\mathbf{u}(t)$ which minimizes the functional $J[\mathbf{x}, \mathbf{u}]$, while satisfying the end-point conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, and the non-holonomic constaints
$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u}).$$

Optimal control problems

Optimization functional

$$J[\mathbf{x}, \mathbf{u}] = \int_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u}) dt$$

Remarks

Note that

- $F(t, \mathbf{x}, \mathbf{u})$ has no dependence on $\dot{\mathbf{u}}$: this is typically because costs depend on the control, not how we change the control, but there might be counter-examples.
- $F(t, \mathbf{x}, \mathbf{u})$ has no dependence on $\dot{\mathbf{x}}$: this is common in control problems, but not universal (we have seen at least one counter example).

Terminal costs

- Sometimes in optimal control we don't fix the end-point $\mathbf{x}(t_1)$, but rather we assign a cost $\phi(t_1, \mathbf{x}(t_1))$ to particular end-points.
- So now we wish to choose a control $\mathbf{u}(t)$ which minimizes the functional

$$J[\mathbf{x}, \mathbf{u}] = \phi(t_1, \mathbf{x}(t_1)) + \int_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u})dt$$

  while satisfying the single end-point condition $x(t_0) = \mathbf{x}_0$, and the non-holonomic constraint $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$.
- $\phi(t_1, \mathbf{x}(t_1))$ is called the terminal cost.

System Terminology

- **linear:** the state equations are a set of linear DEs.
- **autonomous:** time doesn't appear explicitly in the state equations (e.g. in $g(\mathbf{x}, \mathbf{u})$, or $F(\mathbf{x}, \mathbf{u})$).
  - also called time-invariant.
- **terminal cost:** the term $\phi(t_1, \mathbf{x}(t_1))$ is called the terminal cost.
- **controllable:** a solution to the control problem exists.
- **stable:** a stable equilibrium solution to the system DEs exists.
  - often we are interested in problems that are unstable, or we wouldn't really need a control.

Control Terminology

- control (driver or automatic)
  - **planned** (open loop)
  - **feedback** (closed loop) control depends on current state
- type of control
  - movement from A to B
  - continuous operations (maintain equilibrium)
- type of cost functional J
  - minimum time
  - minimum fuel
  - quadratic costs
- admissible controls
  - unbounded / bounded / bang-bang

Cost functional examples

- **minimum time:** choose the fastest possible control

$$J[x,u] = \int_{t_0}^{t_1} dt.$$

- **minimum fuel:** fuel is expended by the controller, and we wish to minimize this

$$J[x,u] = \int_{t_0}^{t_1} |u(t)|dt$$

- **quadratic costs:**

$$J[x,u] = \int_{t_0}^{t_1} \left(x^2(t) + \alpha u^2(t)\right) dt$$

Boundary conditions

- End time t_1: can be fixed or free
- End position $\mathbf{x}(t_1)$: can be fixed or free

In the cases with free boundary conditions, we introduce natural, or transversal boundary conditions.

Example 15.1 Dynamic production

- A producer in purely competitive market
  - A large numbers of independent producers
  - Standardized product, e.g. potatoes
  - Firms are „price takers", i.e. they have no significant control over product price
  - Free entry and exit
  - Free flow of information
- wants to find optimal production path $x(t)$, $0 \leqslant t \leqslant T$.
- production target $x(T) = x_T$
- profit at time t is $\pi(x, \dot{x}, t)$
- maximize profit functional $J[x] = \int\limits_0^T \pi(x, \dot{x}, t)dt$.

Example 15.1 Dynamic production-2

Profit calculation

- quadratic production costs $C_1 = a_1 x^2 + b_1 x + c_1$
  - labor
  - raw materials
- production increase costs $C_2 = a_2 (\dot{x})^2 + b_2 \dot{x} + c_2$
  - new bildings
  - recruiting and training costs
- revenue $r = px$ where p is the constant price per unit
  - $p = const$ due to purely competitive market
- profit at time t is

$$\pi(x, \dot{x}, t) = px - C_1(x) - C_2(\dot{x}).$$

Example 15.1 Dynamic production-3

Problem formulation: maximize total profit

$$J[x] = \int_0^T (px - C_1(x) - C_2(\dot{x}))\, dt$$

subject to $x(0) = 0$ and $x(T) = x_T$.

- notice that the control, and rate of change of state are the same (i.e., $u = \dot{x}$) but we write it as above for simplicity
- autonomous problem
- the control is planned, and has quadratic costs
- admissible controls are unbounded

Example 15.1 Dynamic production-4

Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial \pi}{\partial x} - \frac{d}{dt}\frac{\partial \pi}{\partial \dot{x}} &= 0 \\
p - \frac{\partial C_1}{\partial x} + \frac{d}{dt}\frac{\partial C_2}{\partial \dot{x}} &= 0 \\
p - 2a_1 x - b_1 + \frac{d}{dt}\left[2a_2\dot{x} + b_2\right] &= 0 \\
2a_2\ddot{x} - 2a_1 x + p - b_1 &= 0 \\
\ddot{x} - \frac{a_1}{a_2}x &= \frac{b_1 - p}{2a_2}
\end{aligned}$$

for $a_2 \neq 0$.

Example 15.1 Dynamic production-5

Solution (for $a_1, a_2 \neq 0$)

$$x(t) = Ae^{\sqrt{\frac{a_1}{a_2}}t} + Be^{-\sqrt{\frac{a_1}{a_2}}t} + \frac{b_1 - p}{2a_2}$$

where A and B are determined by the fixed end points $x(0) = x_0$ and $x(T) = x_T$.

This gives the optimal production schedule

- no dependence on c_1 or c_2 (these are constant costs and so shouldn't effect production strategy)
- no dependence on b_2 because this is a linear cost in increasing production, and so occurs regardless of how we increase over time (to get to the final production target $x(T) = x_T$).

Example 15.1 Dynamic production-6

What happens if we make the end point $x(T)$ free, i.e. we don't have a production target at time T?

Then we get a natural boundary condition

$$\left.\frac{\partial \pi}{\partial \dot{x}}\right|_{t=T} = \left.\frac{\partial C_2}{\partial \dot{x}}\right|_{t=T} = 2a_2\dot{x} + b_2\Big|_{t=T} = 0$$

So, rearranging, we get

$$\dot{x}(T) = -\frac{b_2}{2a_2}$$

- constants A and B are determined by end-point conditions $x(0) = 0$ and $\dot{x}(T) = -\frac{b_2}{2a_2}$.

- Production costs

$$C_1 = x^2 + 5x$$

- Production increase costs

$$C_2 = 2\dot{x}^2 + 5\dot{x}$$

- $p = 10$
- $T = 1$
- $x_0 = 0, \quad x_T = 1$