# Calculus of Variations
# Summer Term 2014

Lecture 18

11. Juli 2014

## Purpose of Lesson:

- To introduce Pontryagin's Maximum Principle (PMP)
- To discuss several PMP examples

# Pontryagin's Maximum Principle

Modern optimal control theory often starts from the PMP. It is a simple, coincise condition for an optimal control.

# General control problem

Minimize functional

$$J[\mathbf{x}, \mathbf{u}] = \int_{t_0}^{t_1} F_0(t, \mathbf{x}, \mathbf{u}) dt$$

subject to constraints $\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}, \mathbf{u})$, or more fully,

$$\dot{x}_i = F_i(t, \mathbf{x}, \mathbf{u})$$

- notice no dependence on $\dot{\mathbf{x}}$ in $F_0$
  - this differs from many CoV problems
- no dependence on $\dot{\mathbf{x}}$ in $F_i$ because we rearrange the equations so that derivatives are on the LHS.

# Pontryagin's Maximum Principle (PMP)

Let $\mathbf{u}(t)$ be an admissible control vector that transfers $(t_0, \mathbf{x}_0)$ to a target $(t_1, \mathbf{x}(t_1))$. Let $\mathbf{x}(t)$ be the trajectory corresponding to $\mathbf{u}(t)$.

In order that $\mathbf{u}(t)$ be optimal, it is necessary that there exists $\mathbf{p}(t) = (p_1(t), p_2(t), \dots, p_n(t))$ and a constant scalar $p_0$ such that

- $\mathbf{p}$ and $\mathbf{x}$ are the solution to the canonical system

$$\dot{\mathbf{x}} = \frac{\partial \mathbb{H}}{\partial \mathbf{p}} \quad \text{and} \quad \dot{\mathbf{p}} = -\frac{\partial \mathbb{H}}{\partial \mathbf{x}}$$

- where the Hamiltonian is $\mathbb{H} = \sum\limits_{i=0}^{n} p_i F_i$ with $p_0 = -1$
- $\mathbb{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) \geqslant \mathbb{H}(t, \mathbf{x}, \widehat{\mathbf{u}}, \mathbf{p})$ for all alternate controls $\widehat{\mathbf{u}}$
- all boundary conditions are satisfied

# PMP proof sketch-1

Consider the general problem: minimize functional

$$J[\mathbf{x},\mathbf{u}] = \int_{t_0}^{t_1} F_0(t,\mathbf{x},\mathbf{u})dt$$

subject to constraints

$$\dot{x}_i = F_i(t,\mathbf{x},\mathbf{u}).$$

We can incorporate the constraints into the functional using the Lagrange multipliers $\lambda_i$, e.g.

$$\widehat{J} = \int_{t_0}^{t_1} L(t,\mathbf{x},\dot{\mathbf{x}},\mathbf{u})dt$$

$$= \int_{t_0}^{t_1} F_0(t,\mathbf{x},\mathbf{u})dt + \sum_{i=1}^{n} \lambda_i(t)\left[\dot{x}_i - F_i(t,\mathbf{x},\mathbf{u})\right] dt$$

# PMP proof sketch-2

Given such a function we get (by definition)

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \lambda_i.$$

So we can identify the Lagrange multipliers $\lambda_i$ with the generalized momentum terms $p_i$

1. the $p_i$ are known in economics literature as marginal valuation of $x_i$ or the shadow prices
2. shows how much a unit increment in $x$ at time $t$ contributes to the optimal objective functional $\widehat{J}$
3. the $p_i$ are known in control as co-state variables (sometimes written as $z_i$).

# PMP proof sketch-3

By definition (in previous lecture) the Hamiltonian is

$$\begin{aligned}
\mathbb{H}(t, \mathbf{x}, \mathbf{p}, \mathbf{u}) &= \sum_{i=1}^{n} p_i \dot{x}_i - L(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}, \mathbf{u}) \\
&= \sum_{i=1}^{n} p_i \dot{x}_i - F_0(t, \mathbf{x}, \mathbf{u}) - \sum_{i=1}^{n} \lambda_i(t) \left[\dot{x}_i - F_i(t, \mathbf{x}, \mathbf{u})\right] \\
&= -F_0(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} p_i F_i(t, \mathbf{x}, \mathbf{u})
\end{aligned}$$

because $\lambda_i = p_i$, so the $\dot{x}_i$ terms cancel. The final result is just the Hamiltonian as defined in the PMP.

# PMP proof sketch-4

From previous slide the Hamiltonian can be written

$$\mathbb{H}(t, \mathbf{x}, \mathbf{p}, \mathbf{u}) = -F_0(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} p_i F_i(t, \mathbf{x}, \mathbf{u})$$

which is the Hamiltonian defined in the PMP. Then the canonical E-L equations (Hamilton's equations) are

$$\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dx_i}{dt} \quad \text{and} \quad \frac{\partial \mathbb{H}}{\partial x_i} = -\frac{dp_i}{dt}.$$

Note that the equations $\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dx_i}{dt}$ just revert to

$$F_i(t, \mathbf{x}, \mathbf{u}) = \dot{x}_i$$

which are just the system equations.

# PMP proof sketch-5

Finally, note that Hamilton's equations above only relate $x_i$ and $p_i$. What about equations for $u_i$?

Take the conjugate variable to be $z_i$, and we get (by definition) that

$$z_i = \frac{\partial L}{\partial \dot{u}_i} = 0$$

and the second of Hamilton's equations is therefore

$$\frac{\partial \mathbb{H}}{\partial u_i} = -\frac{dz_i}{dt} = 0$$

which suggests a stationary point of $\mathbb{H}$ WRT $u_i$.

In fact we look for a maximum (and note this may happen on the bounds of $u_i$).

# PMP Example: plant growth

### Example 18.1 (Plant growth-1)

Plant growth problem:

- market gardener wants to plants to grow toa fixed height 2 within a fixed window of time [0, 1]
- can supplement natural growth with lights (at night)
- growth rate dictates
$$\dot{x} = 1 + u$$
- cost of lights
$$J[u] = \int_0^1 \frac{1}{2} u^2 dt$$

# PMP Example: plant growth

**Example 18.1 (Plant growth-2)**

Minimize

$$J[u] = \int_0^1 \frac{1}{2} u^2 dt$$

subject to $x(0) = 0$ and $x(1) = 2$ and

$$\dot{x} = F_1(t, x, u) = 1 + u.$$

Hamiltonian is

$$\begin{aligned} \mathbb{H} &= -F_0(t, x, u) + pF_1(t, x, u) \\ &= -\frac{1}{2}u^2 + p\,(1 + u)\,. \end{aligned}$$

# PMP Example: plant growth

### Example 18.1 (Plant growth-3)

Hamiltonian is

$$\mathbb{H} = -\frac{1}{2}u^2 + p\,(1+u)\,.$$

Canonical equations

$$\frac{\partial \mathbb{H}}{\partial p} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial \mathbb{H}}{\partial x} = -\frac{dp}{dt}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$1 + u = \dot{x} \qquad\qquad 0 = -\dot{p}$$

LHS $\Rightarrow$ system DE

RHS $\Rightarrow$ $\dot{p} = 0$ means that $p = c_1$ where $c_1$ is a constant.

# PMP Example: plant growth

**Example 18.1 (Plant growth-4)**

Maximum principle requires $\mathbb{H}$ be a maximum, for which

$$\frac{\partial \mathbb{H}}{\partial u} = -u + p = 0.$$

So $u = p$, and $\dot{x} = 1 + u$ so

$$x = (1 + c_1)\, t + c_2.$$

The solution which satisfies $x(0) = 0$ and $x(1) = 2$ is

$$x = 2t.$$

So $u = c_1 = 1$, and the optimal cost is $\frac{1}{2}$.

# PMP and natural boundary conditions

Typically we fix $t_0$ and $\mathbf{x}(t_0)$, but often the right-hand boundary condition is not fixed, so we need natural boundary conditions.

Here, they differ from traditional CoV problems in two respects:

- The terminal cost $\phi$
- The function $F_0$ is not explicitly dependent on $\dot{x}$.

The resulting natural boundary conditions are

$$\sum_i \left( \frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} + \left( \frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0$$

for all allowed $\delta x_i$ and $\delta t$.

# PMP and natural boundary conditions

The resulting natural boundary condition is

$$\sum_i \left( \frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} + \left( \frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0.$$

Special cases

- when $t_1$ is fixed and $\mathbf{x}(t_1)$ is completely free we get

$$\left( \frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} = 0, \quad \forall i$$

- when $\mathbf{x}(t_1)$ is fixed, $\delta x_i = 0$, and we get

$$\left( \frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0.$$

# Example: stimulated plant growth

Example 18.2 (Stimulated plant growth-1)

Plant growth problem:

- market gardener wants to plants to grow as much as possible within a fixed window of time $[0, 1]$
- supplement natural growth with lights as before
- growth rate dictates $\dot{x} = 1 + u$
- cost of lights

$$J[u] = \int_0^1 \frac{1}{2} u^2(t) dt$$

- value of crop is proportional to the height

$$\phi(t_1, \mathbf{x}(t_1)) = x(t_1).$$

# Plant growth problem statement

Example 18.2 (Stimulated plant growth-2)

Write as a minimization problem

$$J[x, u] = -x(t_1) + \int_0^1 \frac{1}{2} u^2 dt$$

subject to $x(0) = 0$, and

$$\dot{x} = 1 + u.$$

- the terminal cost doesn't affect the shape of the solution
- but we need a natural end-point condition for $t_1$.

# Plant growth: natural BC

Example 18.2 (Stimulated plant growth-3)

The problem is solved as before, but we write the natural boundary condition at $x = t_1$ as

$$\left(\frac{\partial \phi}{\partial x_i} + p_i\right)\Bigg|_{t=t_1} = 0, \qquad \forall i$$

which reduces to

$$-1 + p\big|_{t=t_1} = 0.$$

Given $p$ is constant, this sets $p(t) = 1$, and hence the control $u = 1$ (as before).

# Autonomous problems

Autonomous problems have no explicit dependence on $t$.

- time invariance symmetry
- hence $\mathbb{H}$ is constant along the optimal trajectory
- if the end-time is free (and the terminal cost is zero) then the transversality conditions ensure $\mathbb{H} = 0$ along the optimal trajectory.

# PMP example: Gout

## Example 18.3 (Gout-1)

Optimal treatment of Gout:

- disease characterized by excess of uric acid in blood
  - define level of uric acid to be $x(t)$
  - in absence of any control, tends to 1 according to

$$\dot{x} = 1 - x$$

- drugs are available to control disease (control $u$)

$$\dot{x} = 1 - x - u$$

  - aim to reduce $x$ to zero as quickly as possible
  - drug is expensive, and unsafe (side effects)

# PMP example: Gout

**Example 18.3 (Gout-2)**

Formulation: minimize

$$J[u] = \int_0^{t_1} \frac{1}{2}\left(k^2 + u^2\right) dt$$

given constant $k$ that measures the relative importance of the drugs cost vs the terminal time.

End-conditions are $x(0) = 1$, and we wish $x(t_1) = 0$, with $t_1$ free. The constraint equation is

$$\dot{x} = 1 - x - u,$$

Hamiltonian

$$\mathbb{H} = -\frac{1}{2}\left(k^2 + u^2\right) + p\,(1 - x - u)\,.$$

# PMP example: Gout

## Example 18.3 (Gout-3)

Canonical equations

$$\frac{\partial \mathbb{H}}{\partial p} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial \mathbb{H}}{\partial x} = -\frac{dp}{dt}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$1 - x - u = \dot{x} \qquad\qquad -p = -\dot{p}$$

LHS $\Rightarrow$ system DE

RHS $\Rightarrow$ $\dot{p} = p$ has solution $p = c_1 e^t$.

Now maximize $\mathbb{H}$ WRT the $u$, i.e., find stationary point

$$\frac{\partial \mathbb{H}}{\partial u} = -u - p = 0$$

So, $u = -p = -c_1 e^t$.

# PMP example: Gout

## Example 18.3 (Gout-4)

Note

- this is an autonomous problem so $\mathbb{H} = \mathit{const}$
- this is a free end-time problem, so $\mathbb{H} = 0$.

Substitute values of $p$ and $u$ into $\mathbb{H}$ for $t = 0$ (i.e. $p = c_1 = -u$, and $x(0) = 1$), and we get

$$\begin{aligned} \mathbb{H} &= -\frac{1}{2}\left(k^2 + u^2\right) + p\,(1 - x - u) \\ &= -\frac{k^2}{2} - \frac{c_1^2}{2} - c_1^2 \\ &= 0 \end{aligned}$$

and so $c_1 = \pm k$.

# PMP example: Gout

## Example 18.3 (Gout-5)

Finally solve $\dot{x} = 1 - x - u$ where $u = -ke^t$ to get

$$x = 1 - \frac{k}{2}e^t + \frac{k}{2}e^{-t} = 1 - k\sinh t$$

The terminal condition is $x(t_1) = 0$, and so

$$t_1 = \sinh^{-1}(1/k)$$

- when $k$ is small the prime consideration is to use a small amount of the drug, and as $k \to 0$ then $t_1 \to \infty$
  - no optimal for $k = 0$
- when $k$ is large, we want to get to a safe level as fast as possible, so as $k \to \infty$ we get $t_1 \sim 1/k$.