Calculus of Variations
Summer Term 2014

Lecture 19

16. Juli 2014

Purpose of Lesson:

- We collect some simple but instructive remarks about the existence and regularity problems of minimizers.
- We begin by presenting some examples of variationals integrals that have weak C^1-minimizers which are not of class C^2.
- To introduce direct methods

§11. Direct Methods of the Calculus of Variations

The following examples show that there are minimizers which are not smooth solutions of the Euler-Lagrange equations.

Example 19.1

A very trivial example is given by the functional

$$J[u] = \int_{-1}^{1} \left\{ u'(x) - 2|x| \right\}^2 dx$$

which is minimized by the functions

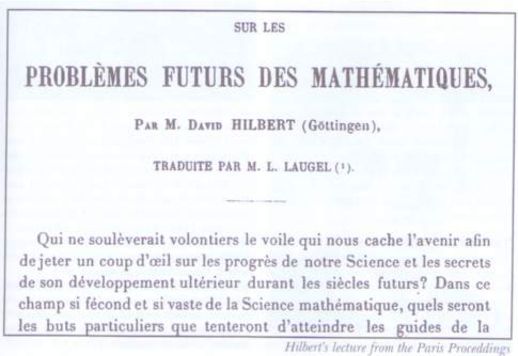
$$u(x) = x|x| + C$$

that are of class C^1 on $[-1, 1]$ but not in C^2 on $(-1, 1)$.

Remarks

- Actually, it is not a-priori clear whether C^1, C^2 or some other function space is the natural setting where a first-order variational problem is to be solved.
- In fact it is part of the problem to define the class of admissible functions where the functionals is to be minimized.
- In general it is not true that every **reasonable problem** (whatever is meant by **reasonable**) has a solution.
- If there exists a solution, one cannot take it for granted that this solution is smooth: there are minimizers which do not satisfy the Euler-Lagrange equation.

Hilbert challenged mathematicians to solve what he considered to be the most important problems of the 20th century in his address to the International Congress of mathematicians in Paris, 1900.



SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (1).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d'atteindre les guides de la

Hilbert's lecture from the Paris Proceedings

David Hilbert around 1900 — Title page from the ICM Paris Proc.

19th *Are the solutions of regular problems in the Calculus of Variations always necessarily analytic?*

20th *Has not every variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also that if need be that the notion of solution be suitably extended?*

These questions have stimulated enormous effort and now things are much better understood.

The existence and regularity theory for elliptic PDE's and variational problems is one of the greatest achievments of 20th century mathematics.

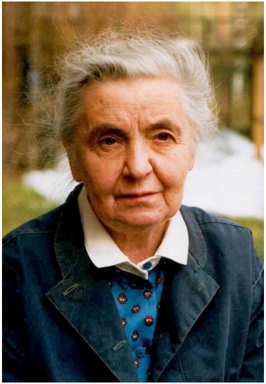
Indirect Methods:

- The classical theory of Calculus of Variations, roughly covers the time from Euler to the end of 19th century is concerned with so-called Indirect Methods.

- The classical indirect method of variational problems is based on the optimistic idea that every minimum problem has a solution.

- In order to determine this solution, one looks for conditions which have to satisfies by a minimizer. An analysis of the necessary conditions often permits one to eliminate many candidates and eventually idetifies a unique solution.

- For example, we only have one solution of the Euler-Lagrange equation satisfying all prescribed conditions. We tempted to infer that this solution is also be a solution to the original minimum problem.



Olga Ladyzhenskaya
(1922-2004)

Nina Uraltseva
(1934 -)

Indirect Methods:

- The above approach is false. We have to prove the existence of minimizer before we could conclude the uniqueness of solutions to the Euler-Lagrange equation.
- Usually, in higher dimension solving the Euler-Lagrange equation is more difficult than finding a minimizer of a functional.

Direct Methods:

- The relevant ideas developed during the 20th century are called Direct Methods.
- An important ingredient here is the introduction of functional analytic techniques.
- In fact, it was the Calsulus of Variations, which gave birth to the theory of Functional Analysis.

The Main Idea of Direct Method:

Consider a minimization problem on some class $\mathcal{A}$ of functions:

$$\min_{u \in \mathcal{A}} J[u]$$

- Suppose there exist a minimizing sequence $\{u_n\}$ in $\mathcal{A}$, i.e.

$$\lim_{n \to \infty} J[u_n] = \min_{u \in \mathcal{A}} J[u] < +\infty.$$

- Suppose we could find an element u_0 in $\mathcal{A}$ such that

$$''\, u_n \to u \,'' \quad \text{as} \quad n \to \infty$$

- Suppose the functional J has some kind of continuity. Therefore

$$\lim_{n \to \infty} J[u_n] = J[u_0].$$

- We conclude that u_0 is a minimizer of the functional J because

$$J[u_0] = \lim_{n \to \infty} J[u_n] = \min_{u \in \mathcal{A}} J[u].$$

Example 19.2 (J is not bounded below)

Let $[0,\pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{u \in C^1\left([0,\pi]\right) : u(0) = u(\pi) = 0\right\}$.

Take

$$J[u] = \int\limits_0^\pi \left[u'^2 - 2u^2\right] dx \qquad \text{and} \qquad u_n(x) = n \sin x \in \mathcal{A}.$$

But $J[u_n] = -\dfrac{\pi n^2}{2} \to -\infty$ as $n \to \infty$. (Also, u_n is unbounded.)

Example 19.3 (No convergent subsequence in minimizing sequence)

Let $[0, \pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{ u \in C^1\left([0, \pi]\right) : u(0) = u(\pi) = 0 \right\}$.

Take

$$J[u] = \int_0^\pi \left(u'^2 - 1\right)^2 dx,$$

$$u_n(x) = \sqrt{\frac{1}{n^2} + \frac{\pi^2}{4}} - \sqrt{\frac{1}{n^2} + \left(x - \frac{\pi}{2}\right)^2} \in \mathcal{A},$$

$$J[u_n] \to 0 = \inf_{v \in \mathcal{A}} J[v].$$

But every subsequence converges to $\frac{\pi}{2} - |x - \frac{\pi}{2}|$ which is not in $\mathcal{A}$.

Example 19.4 (J is not lower semi-continuous)

Let $[0,\pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{u \in C^1([0,\pi]) : u(0) = u(\pi) = 0\right\}$.

Let

$$g(p) = \begin{cases} p^2, \text{ if } p \neq 0; \\ 1, \text{ if } p = 0. \end{cases}$$

Take

$$J[u] = \int_0^{\pi} g(u')dx \quad \text{and} \quad u_n(x) = \frac{1}{n}\sin x \to 0.$$

Then $u_n(x) \in \mathcal{A}$ and $J[u_n] = \dfrac{\pi}{2n^2} \to 0 = I$ as $n \to \infty$ but

$$\pi = J[0] = J[\lim_{n' \to \infty} u_{n'}] > \lim_{n \to \infty} J[u_n] = 0.$$