Calculus of Variations
Summer Term 2014

Lecture 8

23. Mai 2014

Purpose of Lesson:

- To discuss necessary and sufficient conditions for extrema
- To introduce the Legendre condition

§5. Classification of extrema

Introductory remarks

- We have so far typically ignored the issue of classification of extrema.
- Contrariwise, we remember that for simple stationary points we need to look of higher derivatives to see if a stationary point is a maximum, minimum or point of inflection.
- We need an analogous process for extremal curves as well.

Let $f : \mathbb{R} \to \mathbb{R}$. Local extrema have $f'(x) = 0$.

- $f''(x) > 0$ local minima
- $f''(x) < 0$ local maxima
- $f''(x) = 0$ it might be a stationary point of inflection, depending on higher oder derivatives. More precisely, let f be a real-valued, sufficient differentiable function on the interval $I \subset \mathbb{R}$, $x \in I$ and $n \geqslant 1$ an integer.

  If now holds

  $$f'(x) = f''(x) = \cdots = f^n(x) = 0 \quad \text{and} \quad f^{n+1}(x) \neq 0$$

  then, either

  - n is odd and we have a local extremum at x

  or

  - n is even and we have a (local) saddle point at x.

- The E-L equation is a necessary condition.
- The E-L equation is not sufficient.
- Along the extremal curve the functional might have
  - a min, max or stationary point
  - it might be global or local
- We really need to classify extremals
  - Until now we have
    - just assumed it was the minima
    - used analytical insight to understand the solution
    - tested it by inspection
  - We could also compare to alternative curves.

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Examples

- **Physical intuition:** Brachystochrone (or geodesic): we look for a minimum time path. So, we can see that physically there can't be a maximum.

- **Examine the solution:** e.g., consider the functional

$$J[y] = \int_0^1 y'^2 dx$$

conditioned on $y(0) = y(1) = 0$.

The E-L equation gives straight line solutions, e.g., $y = c_1 x + c_2$, and the BCs imply $c_1 = c_2 = 0$, so $y' = 0$. Clearly then $J[y] = 0$, which is the minimum possible value, for an integral of a non-negative function like y'^2.

Examples (cont.)

- **Compare with alternative curves:** Consider the functional

$$J[y] = \int_0^1 \left(xy' + y'^2 \right) dx$$

conditioned on $y(0) = 0$ and $y(1) = 1$.

The E-L equation gives

$$y = -\frac{1}{4}x^2 + C_1 x + C_2$$

and the BCs give $C_1 = 5/4$ and $C_2 = 0$, so the solution is

$$y = \frac{5}{4}x - \frac{1}{4}x^2.$$

Examples (cont.)

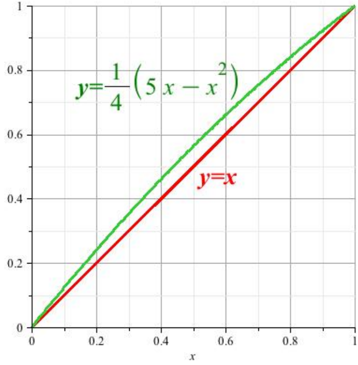
For $y = \frac{5}{4}x - \frac{1}{4}x^2$, we have $y' = \frac{5}{4} - \frac{1}{2}x$, and, consequently,

$$\begin{aligned} J[y] &= \int_0^1 \left[x\left(\frac{5}{4} - \frac{1}{2}x\right) + \left(\frac{5}{4} - \frac{1}{2}x\right)^2 \right] dx \\ &= \int_0^1 \left[\frac{26}{15} - \frac{1}{4}x^2 \right] dx \\ &= \left[\frac{26}{15} - \frac{1}{12}x^3 \right]_0^1 \\ &= \boxed{\frac{71}{48}}. \end{aligned}$$

Examples (cont.)

For the curve $y(x) = x$, we have $y' = 1$, and so the functional is

$$J[y] = \int_0^1 (x+1)^2\,dx = \left[\frac{x^2}{2} + x\right]_0^1 = \boxed{\frac{3}{2}}.$$



Examples (cont.)

Now,

$$\frac{3}{2} > \frac{71}{48},$$

so, we should be looking at a local min.

But, it isn't very formal, or rigorous!

Classification of extrema

- All the methods listed above either
  - Aren't very formal or rigorous
  - Aren't easy to generalize
- Need to develop a means of formal classification
- The secret is by analogy to classification for functions of several variables
  - We need to look at second derivatives
  - Positive defineteness of the Hessian
- The analogy to second derivatives is called the second variation.

Maxima of *n* variables

- If a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ has a local extrema at $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then $\nabla f(\mathbf{x}) = 0$.
- So, we can rewrite Taylor's theorem for small $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ as

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \approx \frac{1}{2}\mathbf{h}^\top H(\mathbf{x})\mathbf{h}$$

- A sufficient condition for the extrema $\mathbf{x}$ to be a local minimum is for the quadratic form

$$Q(h_1, \ldots, h_n) = \mathbf{h}^\top H(\mathbf{x})\mathbf{h} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j$$

to be strictly positive definite.

Definition 8.1

- A quadratic form

$$Q(\mathbf{x}) = \sum_{i,j} a_{ij} x_i x_j = \mathbf{x}^{\mathsf{T}} A \mathbf{x}$$

  is said to be positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

- A quadratic form Q is positive definite iff every eigenvalue of A is greater than zero.

- A quadratic form Q is positive definite if all the principal minors in the top-left corner of A are positive, in other words

$$a_{11} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad \dots$$

Notes on maxima and minima

- Maxima of $f(\mathbf{x})$ is minima of $-f(\mathbf{x})$.
- We need to generalize this for functionals.
- We do this using the second variation.
- Note that even so, we only classify local min and max, the global min and max may occur at the boundary , or at one of the several extrema.

The second variation

- Once again consider the fixed end-point problem, with small perturbations about the extremal curve

$$J[y] = \int_a^b F(x, y, y')dx, \quad y(a) = A, \quad y(b) = B,$$

$$\bar{y} = y + \varepsilon\eta, \quad \eta(a) = \eta(b) = 0.$$

- We take the second derivative of $\phi(\varepsilon) = J[y + \varepsilon\eta]$ with respect to ε, evaluate it $\varepsilon = 0$; that is,

$$\frac{d^2\phi(\varepsilon)}{d\varepsilon^2} = \frac{d^2}{d\varepsilon^2} J[y + \varepsilon\eta]\Big|_{\varepsilon=0}$$
$$= \int_a^b \left[\frac{\partial^2 F}{\partial y^2}\eta^2(x) + 2\frac{\partial^2 F}{\partial y \partial y'}\eta(x)\eta'(x) + \frac{\partial^2 F}{\partial (y')^2}\left(\eta'(x)\right)^2\right] dx$$

The second variation (cont.)

- Note that
$$2\eta\eta' = \frac{d}{dx}\left(\eta^2\right)$$
- So, we can write
$$\int_a^b 2\eta\eta' F_{yy'}dx = \int_a^b \frac{d}{dx}\left(\eta^2\right) F_{yy'}dx$$
$$= \left[\eta^2 F_{yy'}\right]_a^b - \int_a^b \eta^2 \frac{d}{dx}\left(F_{yy'}\right) dx$$

  using integration by parts and the fact that $\eta(a) = \eta(b) = 0$.

The second variation (cont.)

- Now we define the second variation by

$$\delta^2 J[y,\eta] = \int\limits_a^b \left[\frac{\partial^2 F}{\partial y^2}\eta^2(x) + 2\frac{\partial^2 F}{\partial y \partial y'}\eta(x)\eta'(x) + \frac{\partial^2 F}{\partial (y')^2}\left(\eta'(x)\right)^2 \right] dx$$

$$= \int\limits_a^b \left[\eta^2 \left(F_{yy} - \frac{d}{dx}F_{yy'} \right) + \left(\eta'(x)\right)^2 F_{y'y'} \right] dx$$

- This form has the advantage that
  - $\eta^2 \geqslant 0$
  - $\left(\eta'(x)\right)^2 \geqslant 0$
  - after solving the Euler-Lagrange equation we know F and its derivatives.

Classifying extrema

For an extremal curve y to be a local minima, we require

$$\delta^2 J[y, \eta] \geqslant 0$$

for all valid perturbation curves η.

Likewise we get a maxima if $\delta^2 J[y, \eta] \leqslant 0$ for all η and a stationary curve if the second variation changes sign.

- Note that we have already solved the E-L equation and so we know y. Hence we can calculate F_{yy}, $F_{yy'}$ and $F_{y'y'}$ explicitly.
- We still need to ensure $\delta^2 J[y, \eta] \geqslant 0$ for all possible η.

The Legendre condition is a necessary condition for a local minima.

The Legendre condition:

If y is a local minima of the functional $J[y] = \int F(x, y, y')dx$, then

$$F_{y'y'}(x, y, y') \geqslant 0$$

along the extremal curve y.

The Legendre condition (sketch of the proof):

- Remember that F and y are known functions (now), so we know F_{yy}, $F_{yy'}$ and $F_{y'y'}$, explicitly as functions of x.
- Hence we can write the second variation as

$$\delta^2 J[y,\eta] = \int_a^b \left[\eta^2 B(x) + \eta'^2 A(x)\right] dx$$

where

$$A(x) = F_{y'y'}$$
$$B(x) = \left(F_{yy} - \frac{d}{dx} F_{yy'}\right)$$

The Legendre condition (sketch of the proof):

- The proof relies on the fact that we can find functions η such that $|\eta|$ is small, but $|\eta'|$ is large.
- Note that we cannot do the opposite, because $|\eta'|$ small implies that η is smooth, which given the end conditions implies that $|\eta|$ will be small.

Example 8.1 (Mollifier)

$$\eta(x) = \begin{cases} \exp\left(-\dfrac{\gamma}{\gamma^2 - (x-c)^2}\right), & \text{if} \quad x \in [c-\gamma, c+\gamma] \\ 0, & \text{otherwise} \end{cases}$$

Example 8.1 (Mollifier - cont.)

$$\eta(x) = \begin{cases} \exp\left(-\dfrac{\gamma}{\gamma^2-(x-c)^2}\right), & \text{if} \quad x \in [c-\gamma, c+\gamma] \\ 0, & \text{otherwise} \end{cases}$$

$$\eta'(x) = \begin{cases} -\dfrac{2\gamma(x-c)}{\left(\gamma^2-(x-c)^2\right)^2}\exp\left(\dfrac{\gamma}{\gamma^2-(x-c)^2}\right), & \text{if } x \in [c-\gamma, c+\gamma] \\ 0, & \text{otherwise} \end{cases}$$

Ratio of derivative to function is larger for smaller γ.

The Legendre condition (sketch of the proof):

- Given $|\eta|$ small, we can essentially ignore the η^2 terms, and we get only the term

$$\delta^2 J[y,\eta] = \int_a^b \eta'^2 A(x)dx$$

- If A changes sign, then we could choose η to be a mollifier such that it is localized in the part where A is positive, and a mollifier such that it is localized in the part where A is negative.

- The two mollifiers would produce integrals with different signs, and so we would get a change of sign of $\delta^2 J[y,\eta]$, which is what we are trying to avoid.

Example 8.2

Find the minimum of the functional

$$J[y] = \int\limits_0^1 \left(xy' + y'^2 \right) dx$$

conditioned on $y(0) = 0$ and $y(1) = 1$.

The solution is

$$y = \frac{1}{4}\left(5x - x^2\right).$$

Then (from earlier)

$$F(x, y, y') = xy' + y'^2 = \frac{25}{16} - \frac{1}{4}x^2$$

Example 8.2 (cont.)

$$
\begin{aligned}
F(x, y, y') &= xy' + y'^2 \\
F_{y'} &= x + 2y' \\
F_{y'y'} &= 2 > 0
\end{aligned}
$$

Hence Legendre's condition is satisfied, so this could be a local minimum.

Sufficient condition

- various approaches to sufficient conditions
- problem is that we have to get away from pointwise conditions
  - like the Legendre condition
  - pointwise conditions couldn't classify which of two possible arcs of a great circle is the shortest path between two points on a sphere.
- a sufficient condition is the Jacobi condition, but there are others
- still mostly conditions for local minima, so need to do more work

Example 8.3

Find the minimum of the functional

$$J[y] = \int_0^1 \left(xy' + y'^2 \right) dx$$

So

$$F_{y'y'} = 2$$
$$F_{yy'} = F_{yy} = 0$$

Therefore the second variation

$$\delta^2 J[y,\eta] = \int_0^1 \left[\eta^2 \left(F_{yy} - \frac{d}{dx} F_{yy'} \right) + \eta'^2 F_{y'y'} \right] dx = 2 \int_0^1 \eta'^2 dx \geqslant 0$$

for all η we have a local minimum!