Calculus of Variations
Summer Term 2014

Lecture 9

23. Mai 2014

Purpose of Lesson:

- To consider several problems with inequality constraints

§6. Inequality constraints

We have considered problems with

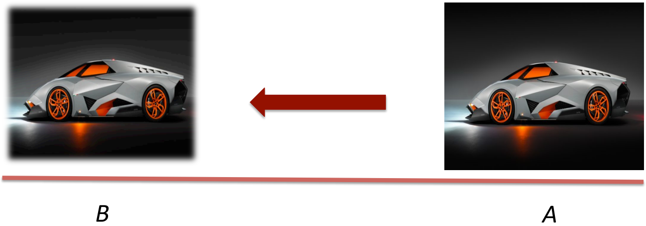
- integral constraints (Dido's problem)
- holonomic constraints (geodesics formulation)
- non-holonomic constraints (problems with higher derivatives)

But we have not considered inequality constraints

Example 9.1: parking a car

Consider the following classic problem:

We want to drive a car/tank from point A to point B as quickly as possible, and at point B the car should be stationary.



Remark

Parking a car seems like a trivial problem:

- in fact this problem appears in other contexts, e.g.
  - automatic positioning of components on a circuit board
  - has to be done frequently (so has to be fast)
  - speed limited by robot, and how delicate the components are
- shortest-time problems are a case of a more general type of problem as well.

http://www.expo21xx.com/automation77/news/2085_robot_mitsubishi/news_default.htm

Example 9.1: parking a car (cont.)

We want to drive a car/tank from point A to point B as quickly as possible, and at point B the car should be stationary.

- Newton's law

$$\text{force} = u = m\ddot{x}$$

- Choose force u that minimizes the time subject to $\dot{x} = 0$ at $t = 0$ and $t = T$, where T is not specified, but rather given by

$$T[u] = \int_A^B dt$$

and it is the functional we wish to minimize.

Example 9.1: parking a car (cont.)

- Note that $\dot{x}(t) = \dfrac{dx}{dt}$ is the car's velocity, so we can write

$$T[x] = \int_A^B dt = \int_{x_A}^{x_B} \frac{1}{\dot{x}} dx$$

- We wish to minimize this functional, subject to the DE constraint that

$$\ddot{x} = \frac{u(t)}{m}$$

where $u(t)$ is the force that we exert, and also subject to

$$\dot{x}(0) = \dot{x}(T) = 0$$

i.e., the car is stationary at the start and finish.

Example 9.1: parking a car (cont.)

- Take $y = \dot{x}$, and we can rewrite the problem as minimize

$$T[y] = \int_{A}^{B} dt = \int_{x_A}^{x_B} \frac{1}{y} dx$$

- We wish to minimize this extremal, subject to the DE constraint that

$$\dot{y} = \frac{u(t)}{m}$$

where $u(t)$ is the control that we exert, and also subject to

$$y(x_A) = y(x_B) = 0.$$

Example 9.1: parking a car (cont.)

- Including the non-holonomic constraint into the problem using a Lagrange multiplier we get

$$\mathcal{H}[y,u] = \int_{x_A}^{x_B} \left[\frac{1}{y} + \lambda \left(\dot{y} - \frac{u(t)}{m} \right) \right] dx$$

subject to

$$y(x_A) = y(x_B) = 0.$$

- The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial h}{\partial \dot{y}} - \frac{\partial h}{\partial y} = 0$$
$$\frac{d}{dt}\frac{\partial h}{\partial \dot{u}} - \frac{\partial h}{\partial u} = 0$$

Example 9.1: parking a car (cont.)

$$\frac{d}{dt}\lambda + \frac{1}{y^2} = 0$$

$$\frac{\lambda}{m} = 0$$

- From the second equation $\lambda = 0$, and so we see that the only viable solutions are $y = \pm\infty$

Example 9.1: parking a car (cont.)

Euler-Lagrange solutions:

- solutions are $y = \pm\infty$
- this requires $u = \pm\infty$ at some points in time
- but in reality we can't exert infinite force
  - i.e., force is bounded

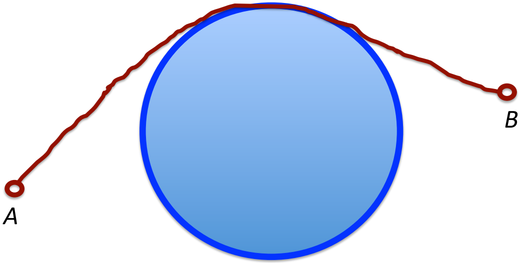
$$|u| \leqslant u_{max}$$

- need to consider optimizing functionals with inequality constraints.
  - similar (in some respects) to min / max functions with inequality constraints
  - min / max is in the interior, or on the boundary

Example 9.2: the shortest path

What is the shortest path, between A and B, avoiding an obstacle.

E.G. what is the shortest path around a lake?



Example 9.2: the shortest path (cont.)

- Find extremals

$$J[y] = \int_{x_0}^{x_1} f(x, y, y')dx$$

sublect to $y(0) = y_0$ and $y(1) = y_1$ and

$$y(x) \geqslant g(x).$$

- Enforce the constraint by taking

$$y(x) = g(x) + z^2(x)$$

In other words introduce a "slack function" $z(x)$, and note that

$$y(x) - g(x) = z^2(x) \geqslant 0.$$

Example 9.2: the shortest path (cont.)

- We have slack function $z(x)$, and constraint $y(x) \geqslant g(x)$ and

$$y = z^2 + g$$
$$y' = 2zz' + g'$$

- Substitute these into the functional and we can change the original functional $J[y]$ for a new one in terms of $J[z]$

$$J[y] = \int_{x_0}^{x_1} f(x, y, y')dx$$

$$J[z] = \int_{x_0}^{x_1} f(x, z^2 + g, 2zz' + g')dx$$

Example 9.2: the shortest path (cont.)

- Given we look for the extremals of

$$J[z]=\int_{x_0}^{x_1} f(x,z^2+g,2zz'+g')dx$$

- The Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dx}\frac{\partial f}{\partial z'}-\frac{\partial f}{\partial z}&=0\\ \frac{d}{dx}\left[2z\frac{\partial f}{\partial y'}\right]-2z\frac{\partial f}{\partial y}-2z'\frac{\partial f}{\partial y'}&=0\\ 2z\frac{d}{dx}\frac{\partial f}{\partial y'}+2z'\frac{\partial f}{\partial y'}-2z\frac{\partial f}{\partial y}-2z'\frac{\partial f}{\partial y'}&=0\\ z\left[\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}\right]&=0\end{aligned}$$

Example 9.2: the shortest path (cont.)

- The Euler-Lagrange equations give

$$z\left[\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}\right]=0$$

for which there are two solutions

  - **Euler areas:** The Euler-Lagrange equations are satisfied
  - **Boundary areas:** $z(x)=0$, so $y(x)=g(x)$ and the curve lies on the boundary

- Analogy: a global minima of function on an interval can happen at stationary point, or at the edges.
- But we can mix the two along the curve y.

Example 9.3: a shortest path around a circular lake

To find the shortest path around a circular lake (radius a, centered at the origin), between the points $(b, 0)$ and $(-b, 0)$ (for $b > a$).

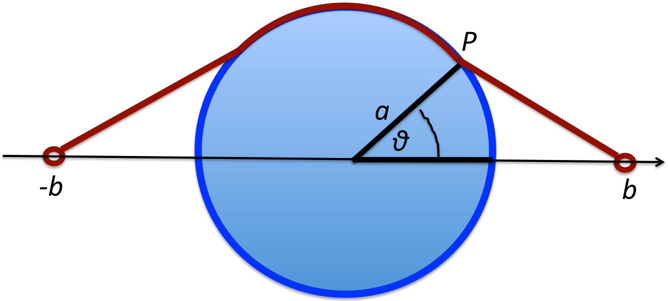
The conditions are

- **Euler areas:** The Euler-Lagrange equations are satisfied, so the curve is a straight line.
- **Boundary areas:** $z(x) = 0$, so $y(x) = g(x)$ and the curve lies on the boundary of the circle.

  We can mix the two along the curve y.

Example 9.3: a shortest path around a circular lake (cont.)

Given the conditions, the solution must look like



i.e. straight lines joining the end-points to a circular arc, where *P*, the point of intersection of the right-hand straight-line, and the circle is at $\left(a\cos(\vartheta), a\sin(\vartheta)\right)$

Example 9.3: a shortest path around a circular lake (cont.)

- The total distance of such a line is

$$\begin{aligned} d(\vartheta) &= 2\sqrt{(b - a\cos\vartheta)^2 + a^2\sin^2\vartheta} + a(\pi - 2\vartheta) \\ &= 2\sqrt{b^2 - 2ab\cos\vartheta + a^2} + a(\pi - 2\vartheta) \end{aligned}$$

- We find the minimum of $d(\vartheta)$, by differentiating WRT ϑ, to get

$$\begin{aligned} d'(\vartheta) &= \frac{2ab\sin\vartheta}{\sqrt{b^2 - 2ab\cos\vartheta + a^2}} - 2a \\ &= 0 \end{aligned}$$

- So,

$$2ab\sin\vartheta = 2a\sqrt{b^2 - 2ab\cos\vartheta + a^2}.$$

Example 9.3: a shortest path around a circular lake (cont.)

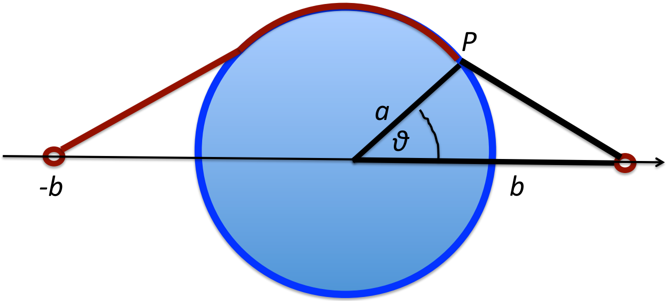
- Dividing both sides by $2a$ we get the condition

$$\begin{aligned} b\sin\vartheta &= \sqrt{b^2 - 2ab\cos\vartheta + a^2} \\ b^2\sin^2\vartheta &= b^2 - 2ab\cos\vartheta + a^2 \\ b^2 - b^2\cos^2\vartheta &= b^2 - 2ab\cos\vartheta + a^2 \\ 0 &= b^2\cos^2\vartheta - 2ab\cos\vartheta + a^2 \\ 0 &= (b\cos\vartheta - a)^2 \end{aligned}$$

- So the result is

$$\cos\vartheta = \frac{a}{b}$$

Example 9.3: a shortest path around a circular lake (cont.)



Think of what we would get if we stretch an elastic band between the two points.