A resonant Lyapunov centre theorem with an application to doubly periodic travelling hydroelastic waves

R. Ahmad* M. D. Groves* D. Nilsson†

Abstract

We present a Lyapunov centre theorem for an antisymplectically reversible Hamiltonian system exhibiting a nondegenerate $1:1$ or $1:-1$ semisimple resonance as a detuning parameter is varied. The system can be finite- or infinite dimensional (and quasilinear) and have a non-constant symplectic structure. We allow the origin to be a 'trivial' eigenvalue arising from a translational symmetry or, in an infinite-dimensional setting, to lie in the continuous spectrum of the linearised Hamiltonian vector field provided a compatibility condition on its range is satisfied.

As an application we show how Kirchgässner's spatial dynamics approach can be used to construct doubly periodic travelling waves on the surface of a three-dimensional body of water (of finite or infinite depth) beneath a thin ice sheet ('hydroelastic waves'). The hydrodynamic problem is formulated as a reversible Hamiltonian system in which an arbitrary horizontal spatial direction is the time-like variable and the infinite-dimensional phase space consists of wave profiles which are periodic (with fixed period) in a second, different horizontal direction. Applying our Lyapunov centre theorem at a point in parameter space associated with a $1:1$ or $1:-1$ semisimple resonance yields a periodic solution of the spatial Hamiltonian system corresponding to a doubly periodic hydroelastic wave.

1 Introduction

1.1 Resonant Hamiltonian systems

A linear dynamical system

$$\dot{v} = Lv$$

for which $L \in \mathbb{R}^{2n\times 2n}$ has a pair of simple, purely imaginary eigenvalues $\pm \mathrm{i}\omega$ has a periodic orbit with frequency ω. The classical *Lyapunov centre theorem* asserts that a (smooth) nonlinear perturbation

$$\dot{v} = Lv + N(v) \tag{1.1}$$

*FR Mathematik, Universität des Saarlandes, Postfach 151150, 66041 Saarbrücken, Germany
†Centre for Mathematical Sciences, Lund University, Box 118, 221 00 Lund, Sweden

of this dynamical system has a family of small-amplitude periodic solutions with frequency near ω provided that (i) it is Hamiltonian or reversible, and (ii) the non-resonance condition that $\mathrm{i}n\omega$ is not an eigenvalue for any integer $n \neq \pm 1$ is satisfied (see Kielhöfer [8, §§I.11.1–I.11.2]). The theorem can be extended to infinite-dimensional, quasilinear systems under the nonresonance condition that $\mathrm{i}n\omega \notin \sigma(L)$ for $n \neq \pm 1$, and furthermore the condition that $0 \notin \sigma(L)$ can also be replaced by a compatibility condition on the range of L (see Iooss [7]).

In this article we consider a parameter-dependent evolutionary equation of the form

$$v_t = L^{\mu_1} v + N^{\mu_1}(v)$$

which is both reversible and Hamiltonian; it may be finite- or infinite-dimensional (and quasilinear). The linear operator L^{μ_1} is supposed to have two pairs $\pm\mathrm{i}\kappa_1^{\mu_1}$, $\pm\mathrm{i}\kappa_2^{\mu_1}$ of simple purely imaginary eigenvalues which depend smoothly upon μ_1 and collide with 'non-zero speed' at $\mu_1 = 0$, that is

$$\kappa_1^0 = \kappa_2^0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1} - \kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0.$$

The collision is semisimple, that is at criticality the eigenvalues $\pm\mathrm{i}\kappa$, where $\kappa = \kappa_1^0 = \kappa_2^0$, are geometrically and algebraically double. We assume the nonresonance condition $\mathrm{i}n\kappa \notin \sigma(L^0)$ for $n = \pm 2, \pm 3, \ldots$ but allow the origin to be a 'trivial' eigenvalue arising from a translational symmetry or, in an infinite-dimensional setting, to lie in the continuous spectrum of L provided a compatibility condition on its range is satisfied. The result is a two-parameter family $\{v(t_1,t_2), \mu_1(t_1,t_2)\}_{0 \leq t_1, t_2 < \varepsilon}$ of $2\pi/(\kappa + \mu_2(t_1,t_2))$-periodic reversible solutions, where $\mu_1(t_1,t_2), \mu_2(t_1,t_2) \to 0$ along with the amplitude of the solutions as $(t_1,t_2) \to (0,0)$.

A classical Hamiltonian system with n degrees of freedom has the form

$$\dot{v} = J\nabla H(v), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where the Hamiltonian H is a smooth real-valued function of $v \in \mathbb{R}^{2n}$ and 0, I denote the $n \times n$ zero and identity matrices; in the above notation $L = J\nabla H_2$, where $H_2(v) = \frac{1}{2}\mathrm{d}^2 H[0](v,v)$. Examining a two-degree-of-freedom Hamiltonian system whose linearisation has two semisimple eigenvalues $\pm\mathrm{i}\kappa$, one finds that $H_2(q_1,q_2,p_1,p_2) = \frac{1}{2}\kappa(q_1^2 + p_1^2) \pm \frac{1}{2}\kappa(q_2^2 + p_2^2)$, the two cases of which are referred to as a $1:1$ and $1:-1$ resonance respectively. (This terminology is also applied to higher-order Hamiltonian systems with no other eigenvalue resonances by restricting to the eigenspaces corresponding to $\pm\mathrm{i}\kappa$.) Periodic solutions of two-degree-of-freedom Hamiltonian systems with semisimple $1:1$ and $1:-1$ resonances were studied by Kummer [9, 10], while the corresponding non-semisimple 'Hamiltonian-Hopf' resonance (in which $\pm\mathrm{i}\kappa$ are geometrically simple and algebraically double eigenvalues) was studied by van der Meer [15].

The dynamical system (1.1) is *reversible* if there exists an involution R (the 'reverser') which anticommutes with L and N. A reversible system has the property that Ru is a solution whenever u is a solution; a solution u is termed *reversible* or *symmetric* if it is invariant under R. Reversible Hamiltonian systems have the property that either $H(Rv) = H(v)$ or $H(Rv) = -H(v)$; these cases are referred to as 'antisymplectic' and 'symplectic' respectively. Small-amplitude periodic solutions of symplectically reversible, n-degree-of-freedom Hamiltonian systems which exhibit a semisimple $1:-1$ resonance were studied by Alomair & Montaldi [1] (see also Buzzi & Lamb [4]) using Lyapunov-Schmidt reduction.

In the present paper we consider parameter-dependent, antisymplectically reversible Hamiltonian systems of the form

$$v_t = J^{\mu_1}(v)\nabla H^{\mu_1}(v) \tag{1.2}$$

which exhibit a semisimple $1:1$ or $1:-1$ resonance in the eigenvalues $\pm\mathrm{i}\kappa$ when $\mu_1 = 0$. The system may be finite- or infinite-dimensional, and $J^{\mu_1}(v)$ is an invertible linear operator which is skew-symmetric with respect to a suitable inner product $\langle\cdot\,,\cdot\rangle$ and, as the notation indicates, is not necessarily constant (see below for a precise statement). We look for periodic solutions of (1.2) with frequency near κ by writing

$$v(t) = u(\tau), \qquad \tau = (\kappa + \mu_2)t$$

and seeking 2π-periodic solutions of the transformed equation. For this purpose we use variational Lyapunov-Schmidt reduction, seeking critical points of the functional

$$S(u,\mu_1,\mu_2) = \frac{1}{2\pi}\int_0^{2\pi} \left\{ -(\kappa+\mu_2)\langle\alpha^{\mu_1}(u), u_\tau\rangle - H^{\mu_1}(u) \right\} \mathrm{d}\tau,$$

where $\alpha^{\mu_1}(v)$ is an anti-derivative of $J^{\mu_1}(v)$, in a suitable 'loop space'. A precise statement of our theorem is given in Section 1.2 below; the proof is presented in Section 2.

Hamiltonian formalism

Let X, Z be real Hilbert spaces, where X is continuously and densely embedded in Z, and Z is equipped with an additional continuous inner product $\langle\cdot\,,\cdot\rangle$ which does not necessarily induce its strong topology. Let $\Lambda_1 \times U$ be a neighbourhood of the origin in $\mathbb{R} \times X$. We regard U as a manifold domain of Z by extending elements of the tangent space $TX|_v \cong X^* \cong X$ of X at the point $v \in U$ to elements of the tangent space $TZ|_v \cong Z^* \cong Z$ of Z at this point. The derivative $\mathrm{d}F^{\mu_1}[v] \in X^*$ of a smooth real-valued function $F^{\mu_1}(v)$ of $(\mu_1, v) \in \Lambda_1 \times U$ has a unique extension $\widetilde{\mathrm{d}F}^{\mu_1}[v] \in Z^*$. We use the same notation for smooth functions $F^{\mu_1}(v)$ of (μ_1, v) with values in X, so that $\widetilde{\mathrm{d}F}^{\mu_1}[v] \in \mathcal{L}(Z)$, and occasionally assume that there exists an adjoint operator $\widetilde{\mathrm{d}F}^{\mu_1}[v]^* \in \mathcal{L}(Z)$ such that

$$\langle\widetilde{\mathrm{d}F}^{\mu_1}[v]^*(v_1), v_2\rangle = \langle v_1, \widetilde{\mathrm{d}F}^{\mu_1}[v](v_2)\rangle$$

for all $(\mu_1, v) \in \Lambda_1 \times U$ and v_1, $v_2 \in Z$. Both $\widetilde{\mathrm{d}F}^{\mu_1}[v]$ and $\widetilde{\mathrm{d}F}^{\mu_1}[v]^*$ are assumed to depend smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$.

Using this framework we make the following definitions and assumptions.

(i) *A (parameter-dependent) k-form on U* is an alternating, bounded, k-linear mapping $Z^k \to \mathbb{R}$ which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$. An *exact symplectic 2-form* $\Omega^{\mu_1}|_v$ on U is a 2-form given by

$$\Omega^{\mu_1}|_v(v_1, v_2) = \langle J^{\mu_1}(v)v_1, v_2\rangle,$$

where $J^{\mu_1}(v)$ is an invertible, skew-symmetric linear mapping in $\mathcal{L}(Z)$ which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$. Furthermore $\Omega^{\mu_1}|_v$ is the exterior derivative of a 1-form $\omega^{\mu_1}|_v$ given by

$$\omega^{\mu_1}|_v(w) = \langle \alpha^{\mu_1}(v), w \rangle,$$

where $\alpha^{\mu_1}(v)$ is an element of Z which depends smoothly upon $(\mu_1, v) \in \Lambda_1 \times U$; in other words

$$\Omega^{\mu_1}|_v(v_1, v_2) = \langle \widetilde{\mathrm{d}\alpha}^{\mu_1}[v](v_1), v_2 \rangle - \langle v_1, \widetilde{\mathrm{d}\alpha}^{\mu_1}[v](v_2) \rangle,$$

so that

$$J^{\mu_1}(v) = \widetilde{\mathrm{d}\alpha}^{\mu_1}[v] - \widetilde{\mathrm{d}\alpha}^{\mu_1}[v]^*$$

(ii) *A (parameter-dependent) Hamiltonian on* U is a smooth real-valued function $H^{\mu_1}(v)$ of $(\mu_1, v) \in \Lambda_1 \times U$ which satisfies $H^{\mu_1}(0) = 0$ and $\mathrm{d}H^{\mu_1}[0] = 0$ for all $\mu_1 \in \Lambda_1$. We assume that its gradient, that is the element $\nabla H^{\mu_1}(v)$ of Z with

$$\widetilde{\mathrm{d}H}^{\mu_1}[v](w) = \langle \nabla H^{\mu_1}(v), w \rangle$$

for all $w \in Z$, exists for all v in a dense subset $\mathcal{D}_\mathrm{H}$ of $\Lambda_1 \times U$ and extends to a smooth function of $(\mu_1, v) \in \Lambda_1 \times U$.

(iii) *The Hamiltonian vector field* $v_\mathrm{H}^{\mu_1}$ of a *Hamiltonian system* $(Z, \Omega^{\mu_1}, H^{\mu_1})$, where Ω^{μ_1} is an exact symplectic 2-form and H^{μ_1} is a Hamiltonian on U, is given by

$$v_\mathrm{H}^{\mu_1}(v) = J^{\mu_1}(v)^{-1} \nabla H^{\mu_1}(v), \qquad (\mu_1, v) \in \mathcal{D}_\mathrm{H};$$

it yields the unique element $v_\mathrm{H}^{\mu_1}(v)$ of Z such that

$$\Omega^{\mu_1}|_v(v_\mathrm{H}^{\mu_1}(v), w) = \widetilde{\mathrm{d}H}^{\mu_1}[v](w)$$

for all $w \in Z$. *Hamilton's equations*

$$v_t = v_\mathrm{H}^{\mu_1}(v), \qquad (\mu_1, v) \in \mathcal{D}_\mathrm{H},$$

determine the orbits generated by the Hamiltonian vector field.

1.2 The main result

Let X, Z be real Hilbert spaces, where X is continuously and densely embedded in Z, and consider the autonomous evolutionary equation

$$v_t = L^{\mu_1} v + N^{\mu_1}(v), \tag{1.3}$$

where $(\mu_1, v) \mapsto N^{\mu_1}(v)$ is a smooth mapping from a neighbourhood $\Lambda_1 \times U$ of the origin in $\mathbb{R} \times X$ into Z which satisfies $N^{\mu_1}(0) = 0$ and $\mathrm{d}N^{\mu_1}[0] = 0$ for all $\mu_1 \in \Lambda_1$, and $L^{\mu_1} : X \subseteq Z \to Z$ is a closed linear operator which depends smoothly upon μ_1. We study equation (1.3) under the following hypotheses.

(H1) Equation (1.3) represents Hamilton's equations

$$v_t = v_{\mathrm{H}}^{\mu_1}(v)$$

for a Hamiltonian system $(Z, \Omega^{\mu_1}, H^{\mu_1})$ with $\mathcal{D}_{\mathrm{H}} = \Lambda_1 \times U$ (in the above nomenclature), so that

$$L^{\mu_1}(v) + N^{\mu_1}(v) = J^{\mu_1}(v)^{-1} \nabla H^{\mu_1}(v)$$

and in particular

$$L^{\mu_1} v = J^{\mu_1}(0)^{-1} \nabla H_2^{\mu_1}(v),$$

where $H_2^{\mu_1}(v) = \frac{1}{2} \mathrm{d}^2 H^{\mu_1}[0](v, v)$ (the part of H^{μ_1} which is homogeneous of degree 2 in v).

(H2) Equation (1.3) is reversible: both L^{μ_1} and N^{μ_1} anticommute with an involution $R \in \mathcal{L}(X) \cap \mathcal{L}(Z)$. This *reverser* R satisfies

$$H^{\mu_1}(Rv) = H^{\mu_1}(v), \quad R^* \alpha^{\mu_1}(Rv) = -\alpha^{\mu_1}(v), \quad R^* J^{\mu_1}(Rv) R = -J^{\mu_1}(v)$$

for all $(\mu_1, v) \in \Lambda_1 \times U$.

There are also spectral hypotheses on L^{μ_1}.

(H3) The linear operator L^{μ_1} has two pairs $\pm \mathrm{i}\kappa_1^{\mu_1}$, $\pm \mathrm{i}\kappa_2^{\mu_1}$ of purely imaginary eigenvalues with linearly independent eigenvectors $e_1^{\mu_1}$, $\bar{e}_1^{\mu_1}$, $e_2^{\mu_1}$, $\bar{e}_2^{\mu_1}$, all of which depend smoothly upon μ_1. Furthermore $\bar{e}_1^0 = Re_1^0$, $\bar{e}_2^0 = Re_2^0$ and

$$\kappa_1^0 = \kappa_2^0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu_1}(\kappa_1^{\mu_1} - \kappa_2^{\mu_1})\Big|_{\mu_1=0} \neq 0.$$

For notational simplicity we henceforth abbreviate L^0, e_1^0 and e_2^0 to respectively L, e_1 and e_2, and define $\kappa := \kappa_1^0 = \kappa_2^0$.

(H4) The origin is one of

(i) a point of the resolvent set of L,

(ii) a point of the continuous spectrum of L,

(iii) a geometrically simple, algebraically double eigenvalue of L with eigenvector f_1 and generalized eigenvector f_2 (possibly embedded in continuous spectrum), where $Rf_1 = -f_1$ and $Rf_2 = f_2$.

Here (ii) and (iii) entail further spectral hypotheses (see (H7) and (H8) below).

(H5) The imaginary number $\mathrm{i}k\kappa$ belongs to the resolvent set of L for each $k \in \mathbb{Z} \backslash \{0, -1, 1\}$.

(H6) The linear operator L satisfies

$$\left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z \to Z} \lesssim \frac{1}{|k|}, \qquad \left\|(\mathrm{i}k\kappa I - L)^{-1}\right\|_{Z \to X} \lesssim 1$$

for all $k \in \mathbb{Z} \backslash \{0, -1, 1\}$.

(H7) The zero eigenvalue (if present) is 'trivial' in the following sense: writing $u \in U$ as $u = qf_1 + w$ with $w \in \{f_1\}^\perp$, one finds that $J^{\mu_1}(u)$ and $H^{\mu_1}(u)$ do not depend upon q.

(H8) Suppose that the equation

$$Lu^\dagger = J^0(0)^{-1}(I - \Pi_0)N^\star(u, \mu_1, \mu_2),$$

where

$$N^\star(u, \mu_1, \mu_2) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Big(J^{\mu_1}(u)\big((\kappa+\mu_2)J^{\mu_1}(u)u_\tau - L^{\mu_1}u - N^{\mu_1}(u)\big) + J^0(0)Lu \Big)\, \mathrm{d}\tau,$$

has a unique solution $u^\dagger \in (I - \Pi_0)X$ which depends smoothly upon $u \in \mathcal{U}$, $\mu_1 \in \Lambda_1$ and $\mu_2 \in \Lambda_2$, where Π_0 is the orthogonal projection of X onto $\operatorname{span}\{f_1, f_2\}$ and

$$\mathcal{U} = \{u \in H^1_{\mathrm{per}}(\mathbb{R}, Z) \cap L^2_{\mathrm{per}}(\mathbb{R}, X) : u(\tau) \in U \text{ for all } \tau \in \mathbb{R}\}.$$

Remark 1.1.

(i) Hypothesis (H8) is meaningful only if the origin lies in the continuous spectrum of L or is an eigenvalue embedded in continuous spectrum, since it is automatically satisfied if the origin lies in the resolvent set of L or is an isolated eigenvalue of L.

(ii) If $J^{\mu_1}(u)$ is constant and $\Pi_0 = 0$, then

$$N^\star(u, \mu_1, \mu_2) = -\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} N^{\mu_1}(u)\, \mathrm{d}\tau,$$

so that hypothesis (H8) reduces to the condition used by Iooss [7].

Theorem 1.2. *Under hypotheses (H1)–(H8) there exist $\varepsilon > 0$ and a smooth, two-parameter branch $\{v(t_1,t_2), \mu_1(t_1,t_2)\}_{0 \le t_1,t_2 < \varepsilon}$ of $2\pi/(\kappa+\mu_2(t_1,t_2))$-periodic reversible solutions to equation* (1.3) *in $H^1_{\mathrm{loc}}(\mathbb{R}, Z) \cap L^2_{\mathrm{loc}}(\mathbb{R}, X)$. The rescaled function $v(t) = u(\tau)$, $\tau = (\kappa+\mu_2)t$ satisfies $\|u(t_1,t_2)\|_{\mathcal{Z}} \to 0$, while $\mu_1(t_1,t_2), \mu_2(t_1,t_2) \to 0$ as $(t_1,t_2) \to (0,0)$.*

Theorem 1.2 is proved in Section 2.

1.3 Hydroelastic waves

In Section 3 we introduce the hydrodynamic problem for travelling waves on the surface of a three-dimensional body of water beneath a thin ice sheet modelled using the Cosserat theory of hyperelastic shells (Plotnikov & Toland [14]). The fluid is bounded below by a rigid horizontal bottom $\{x_2 = -h\}$ (the cases $h < \infty$ and $h = \infty$ are referred to as 'finite depth' and 'infinite depth') and above by a free surface $\{x_2 = \eta(x_1, x_3)\}$ (in a frame of reference following the wave with constant speed c in the x_1 direction); there is no cavitation between this surface and the ice sheet. The hydrodynamic problem is formulated in terms of η and an Eulerian velocity potential ϕ in dimensionless form in Section 3; the govering equations (3.1)–(3.4) depend upon two dimensionless parameters

$$\beta = \left(\frac{D}{\rho g h^4}\right)^{1/4} \geq 0, \qquad \gamma = \left(\frac{c^8 \rho}{D g^3}\right)^{1/8} > 0,$$

where D, ρ and g are respectively the coefficient of flexural rigidity for the ice sheet, the density of the fluid and the acceleration due to gravity (see Guyenne and Parau [6]). The dimensionless fluid domain is $\{-\frac{1}{\beta} < x_2 < \eta(x_1, x_2)\}$, so that the limit $\beta \to 0$ corresponds to 'infinite depth'.

We consider waves which are periodic with periods p_1 and p_2 in two arbitrary horizontal directions x and z which form (different) angles θ_1, $\theta_2 \in [0, \pi)$ with the x_1-axis respectively, so that

$$x = \csc(\theta_2 - \theta_1)(x_1 \sin\theta_2 - x_3 \cos\theta_2), \qquad z = \csc(\theta_1 - \theta_2)(x_1 \sin\theta_1 - x_3 \cos\theta_1)$$

(see Figure 1). To this end we seek solutions of the governing equations of the form

$$\eta(x_1, x_3) = \tilde{\eta}(\tilde{x}, \tilde{z}),$$

where

$$\tilde{x} = x_1 \sin\theta_2 - x_3 \cos\theta_2, \qquad \tilde{z} = \nu(x_1 \sin\theta_1 - x_3 \cos\theta_1)$$

with $\nu = 2\pi/p_2$ and $\tilde{\eta}$ is 2π-periodic in $\tilde{z}$. We proceed by formulating the hydroelastic problem as a reversible Hamiltonian system in which the horizontal spatial direction $\tilde{x}$ plays the role of the time-like variable ('spatial dynamics'), working in a phase space of functions which are 2π-periodic in $\tilde{z}$. By construction a periodic (in 'time') solution of the evolutionary system, corresponds to a surface wave which is periodic in both $\tilde{x}$ and $\tilde{z}$ and is found using a Lyapunov centre theorem (although care is required in interpreting such solutions; see below).

To formulate the hydroelastic equations as an evolutionary system, we exploit the observation that they follow from a variational principle (a modified version of the classical variational principle for water waves introduced by Luke [12]). We treat the variational functional as an action functional in which a density is integrated over the $\tilde{x}$ direction and perform a Legendre transform (which is actually higher order due to the presence of second-order derivatives in the density) to derive a formulation of the hydrodynamic problem as an infinite-dimensional, reversible Hamiltonian system in which $\tilde{x}$ is the time-like variable. Although this procedure is formal, it delivers a candidate for a formulation of the hydrodynamic problem as an evolutionary system whose mathematical correctness is readily confirmed a posteriori; full details are given in Section 3. Finally, we introduce a bifurcation parameter μ_1 by writing $\nu = \nu_0 + \mu_1$, where ν_0 is a reference value for ν (see below) and use a change of variable to linearise a nonlinear boundary condition emerging from the Legrendre transform. The result is a system of the form

$$\hat{v}_x = L^{\mu_1}\hat{v} + N^{\mu_1}(\hat{v}) \tag{1.4}$$

which is amenable to Theorem 1.2, satisfying (H1) and (H2) by construction.

A purely imaginary eigenvalue is of $L := L^0$ with corresponding eigenvector in the kth Fourier mode (where $(k, s) \neq (0, 0)$) corresponds to a linear hydroelastic wave of the form $\eta(\tilde{x}, \tilde{z}) = \eta_{s,k} e^{i\ell_1\tilde{x} + i\ell_2\tilde{z}}$ with

$$\ell_1 = s \sin\theta_2 + \nu_0 k \sin\theta_1, \tag{1.5}$$

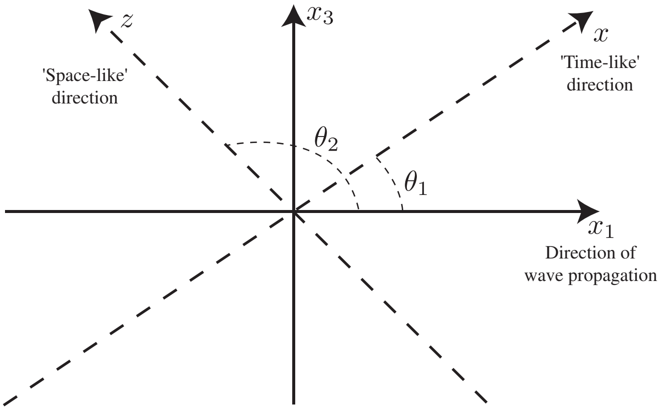

Figure 1: In the spatial dynamics formulation of the hydrodynamic problem the x and z directions are treated as respectively 'time-like' and 'space like'.

$$\ell_2 = -s\cos\theta_2 - \nu_0 k \cos\theta_1, \tag{1.6}$$

and a solution of this kind exists if and only if ℓ_1 and ℓ_2 satisfy the dispersion relation

$$\mathcal{D}(\ell_1, \ell_2) := (1 + (\ell_1^2 + \ell_2^2)^2)\sqrt{\ell_1^2 + \ell_2^2}\tanh\left(\beta^{-1}\sqrt{\ell_1^2 + \ell_2^2}\right) - \gamma^2\ell_1^2 = 0.$$

A mode k purely imaginary eigenvalue is therefore corresponds to an intersection in the (ℓ_1, ℓ_2)-plane of the dispersion curve

$$C_{\mathrm{dr}} = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \setminus \{(0,0)\} : \mathcal{D}(\ell_1, \ell_2) = 0\}$$

with the straight line S_k defined by equations (1.5), (1.6). (The solution $(\ell_1, \ell_2) = (0,0)$ of $\mathcal{D}(\ell_1, \ell_2) = 0$ is excluded since it corresponds to $(k, s) = (0, 0)$.) The dispersion curve is sketched in Figure 2(a). The (β, γ)-parameter plane is divided into three regions in which C_{dr} has respectively zero, one, and two nontrivial bounded branches; the delineating curves D_1 and D_2 are given explicitly in Section 4.1. The central region is in fact each divided into two subregions, at the mutual boundary of which the qualitative shape of the branches changes, namely, from convex to nonconvex.

A point of intersection of S_k and C_{dr} corresponds to a purely imaginary mode k eigenvalue is; its imaginary part is the value of the parameter s at the point of intersection, that is, the value of S_0 in the (S_0, T)-coordinate system at the intersection, where

$$T = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \ : \ \ell_1 = \sin\theta_1\, a,\ \ell_2 = -\cos\theta_1\, a,\ a \in \mathbb{R}\}$$

(see Figure 2(b)). The geometric multiplicity of the eigenvalue is is given by the number of distinct lines in the family $\{S_k\}$ which intersect C_{dr} at this parameter value, and a tangent intersection between S_k and C_{dr} indicates that each eigenvector in mode k has an associated Jordan chain of length at least 2. Notice that the sets $S_k \cap C_{\mathrm{dr}}$ and $S_{-k} \cap C_{\mathrm{dr}}$ have the same cardinality: the purely imaginary number is is a mode k eigenvalue if and only if the purely imaginary number $-$is is a mode $-k$ eigenvalue.

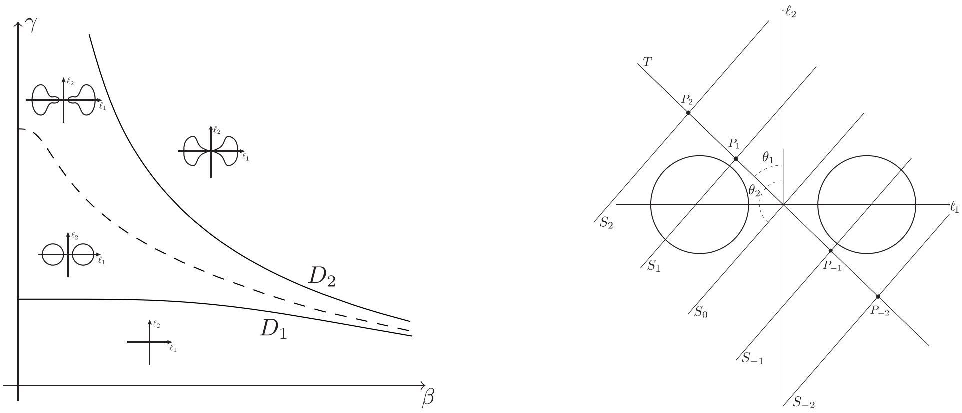

Figure 2: (a) Shape of the dispersion curve C_{dr} in the (β, γ)-parameter plane. (b) Position of the lines S_k and the dispersion curve C_{dr} in the (ℓ_1, ℓ_2)-plane; the lines S_0 and T form rangles θ_2 and θ_1 respectively with the positive ℓ_2 axis, and the line S_k intersects the line T at the point P_k.

Let us now consider how to apply a Lyapunov centre theorem to equation (1.4). A first attempt might be to choose β, γ and θ_2 such that S_0 does not intersect C_{dr} and θ_1 such that S_1 and S_{-1} intersect C_{dr} in points with coordinates (s, ν_0), (t, ν_0) and $(-s, \nu_0)$, $(-t, \nu_0)$ in the (S_0, T)-coordinate system respectively, while S_k does not intersect C_{dr} for $k = \pm 2, \pm 3, \ldots$ (see Figure 2(b)). In this configuration L has simple mode 1 eigenvalues $\mathrm{i}s$, $\mathrm{i}t$ with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{1,t}\mathrm{e}^{\mathrm{i}z}$ and mode -1 eigenvalues $-\mathrm{i}s$, $-\mathrm{i}t$ with eigenvectors $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$, $\hat{v}_{-1,t}\mathrm{e}^{-\mathrm{i}z}$. Assuming that $0 < t < s$ (and neglecting any spectrum at the origin for the moment), one finds that the purely imaginary eigenvalues $\pm\mathrm{i}s$ satisfy the non-resonance condition in a standard Lyapunov centre theorem, an infinite-dimensional version of which therefore gives a periodic solution of (1.4) with period near $2\pi/s$. However this approach does not yield a genuinely three-dimensional wave: at the linear level the solution takes the form $\hat{v} = \mathrm{Re}(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}sx}\mathrm{e}^{\mathrm{i}z}) = \mathrm{Re}(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}(sx+z)})$, which depends upon the single spatial direction $sx + z$. Note that Groves & Haragus [5] and Bagri & Groves [2], while correctly elucidating the use of spatial dynamics and Lyapunov centre theory to construct doubly periodic water waves, actually detect waves of this kind, often referred to as '$2\frac{1}{2}$-dimensional waves'.

A more promising approach is to choose β, γ and θ_2 such that S_0 does not intersect C_{dr} and ν_0 and θ_1 such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, while S_k does not intersect C_{dr} for $k = \pm 2, \pm 3, \ldots$ (see Figure 3). In this configuration L exihibits a $1:1$ or $1:-1$ resonance: it has two mode 1 eigenvalues $\pm\mathrm{i}s$, two mode -1 eigenvalues $\pm\mathrm{i}s$, so that $\pm\mathrm{i}s$ are geometrically and algebraically double eigenvalues of L with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$ and $\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z}$. Under the assumption that the other hypotheses are satisfied, Theorem 1.2 gives a periodic solution of (1.4) with period near $2\pi/s$ which yields a genuinely three-dimensional wave: at the linear level the solution takes the form $\hat{v} = (\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}sx}\mathrm{e}^{\mathrm{i}z} + \hat{v}_{1,-s}\mathrm{e}^{-\mathrm{i}sx}\mathrm{e}^{-\mathrm{i}z})$, which cannot be reduced to a function of a single spatial direction. (One can of course apply this idea in any mode, assuming that $\pm\mathrm{i}s$ are both mode k and mode $-k$ eigenvalues which do not resonate with any other purely imaginary eigenvalues.)

In Section 4 we apply Theorem 1.2 to the spatial dynamics formulation (1.4) of the hydro-elastic problem in the eigenvalue scenarios shown in Figure 3, studying the purely imaginary

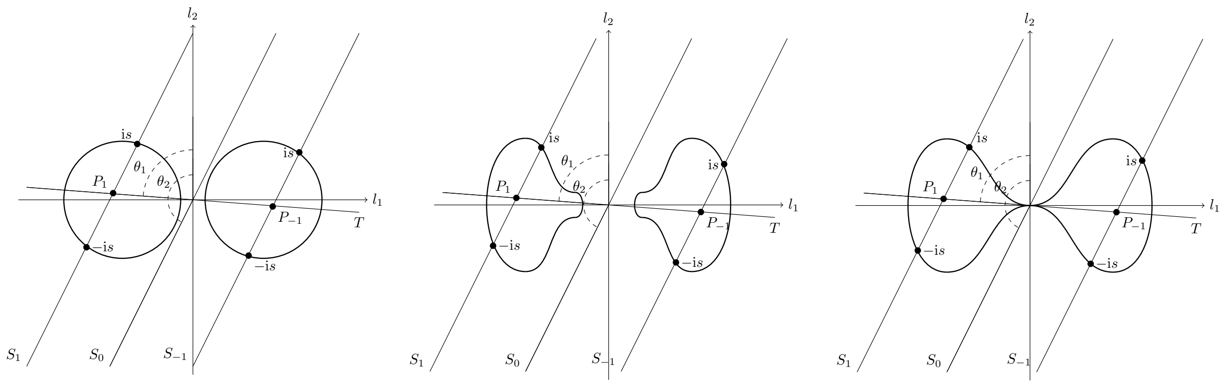

Figure 3: Scenarios of interest.

spectrum of L in detail in Section 4.1. Hypotheses (H3), (H5) and (H6) are readily verified, while (H4)(iiii) (the origin is a double eigenvalue of L) and (H4)(ii) (the origin is a point of the continuous spectrum of L) arise in the cases $\beta > 0$ and $\beta = 0$ respectively. In the former case we find that the zero eigenvalue is trivial in the sense of (H7) and an isolated spectral point of L, so that (H8) is also satisfied. The additional verification of (H8) in the case $\beta = 0$ is undertaken in Section 4.3. Altogether we establish the following result.

Theorem 1.3. *Choose β, γ and θ_2 such that S_0 does not intersect $C_{\rm dr}$ and ν_0 and θ_1 such that S_1 and S_{-1} each intersect $C_{\rm dr}$ in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, while S_k does not intersect $C_{\rm dr}$ for $k = \pm 2, \pm 3, \ldots$ (see Figure 3). There exist $\varepsilon > 0$ and a two-parameter branch $\{(\phi, \eta)(t_1, t_2)\}_{0 \leq t_1, t_2 < \varepsilon}$ of doubly periodic solutions of* (3.1)–(3.4) *with periods $2\pi/(s + \mu_2(t_1, t_2))$, $2\pi/(\nu_0 + \mu_1(t_1, t_2))$ in the variables $x_1 \sin\theta_2 - x_3 \cos\theta_2$ and $x_1 \sin\theta_1 - x_3 \cos\theta_1$ respectively.*

We also derive the following 'inverse' result which shows that (under a nonresonance condition) one can find a family of doubly periodic solutions which are small perturbations of any given periodic cell; these solutions have a fixed dimensionless wave speed γ and fixed periodic directions.

Theorem 1.4. *Choose β, s, ν_0 and $\theta_2 - \theta_1$ arbitrarily. There exist θ_1 and γ such that S_1 and S_{-1} each intersect $C_{\rm dr}$ in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, so that Theorem 1.3 holds under the additional hypothesis that S_k does not intersect $C_{\rm dr}$ for $k \neq \pm 1$.*

2 Proof of the main result

In this section we prove Theorem 1.2, working in the framework set out in Section 1.2 and under the hypotheses (H1)–(H8) given there. We look for periodic solutions of (1.3) with frequency near κ by writing

$$v(t) = u(\tau), \qquad \tau = (\kappa + \mu_2)t,$$

where μ_2 lies in a neighbourhood Λ_2 of the origin in $\mathbb{R}$, and seeking 2π-periodic solutions of the transformed equation

$$(\kappa + \mu_2)u_\tau = L^{\mu_1}u + N^{\mu_1}(u). \tag{2.1}$$

To this end we introduce the function spaces

$$\mathcal{X} := H^1_{\mathrm{per}}(\mathbb{R}, Z) \cap L^2_{\mathrm{per}}(\mathbb{R}, X),$$
$$\mathcal{Z} := L^2_{\mathrm{per}}(\mathbb{R}, Z),$$

equipping $\mathcal{Z}$ with the continuous scalar product

$$(\cdot\,,\cdot) = \frac{1}{2\pi}\int_0^{2\pi} \langle \cdot\,,\cdot\rangle$$

and noting that elements $w \in \mathcal{Z}$ can be expanded in Fourier series

$$w(\tau) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}[w]_k \mathrm{e}^{\mathrm{i}k\tau}, \qquad [w]_k = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} w(\tau)\mathrm{e}^{-\mathrm{i}k\tau}\,\mathrm{d}\tau \in Z.$$

We seek 2π-periodic solutions of (2.1) by studying the function $F : \mathcal{U} \times \Lambda_1 \times \Lambda_2 \mapsto \mathcal{Z}$ defined by

$$F(u, \mu_1, \mu_2) = (\kappa + \mu_2)J^{\mu_1}(u)u_\tau - \nabla H^{\mu_1}(u),$$

where

$$\mathcal{U} = \{u \in \mathcal{X} : u(\tau) \in U \text{ for all } \tau \in \mathbb{R}\}.$$

The equation

$$F(u, \mu_1, \mu_2) = 0, \tag{2.2}$$

has a variational characterisation.

Proposition 2.1. *Equation* (2.2) *is the Euler-Lagrange equation for the action functional* $S : \mathcal{U} \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$ *given by*

$$S(u, \mu_1, \mu_2) = \frac{1}{2\pi}\int_0^{2\pi} \Big\{ -(\kappa + \mu_2)\langle \alpha^{\mu_1}(u), u_\tau\rangle - H^{\mu_1}(u)\Big\}\,\mathrm{d}\tau.$$

Furthermore S *is invariant with respect to the translation* $T_\theta : u(\tau) \mapsto u(\tau+\theta)$ *and reversing operation* $T : u(\tau) \mapsto (Ru)(-\tau)$.

Proof. The first assertion follows from the calculation

$$\begin{aligned}
&\mathrm{d}_1 S[u, \mu_1, \mu_2](v)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big\{ -(\kappa+\mu_2)\big(\langle \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](v), u_\tau\rangle + \langle \alpha^{\mu_1}(u), v_\tau\rangle\big) - \langle \nabla H^{\mu_1}(u), v\rangle \Big\}\,\mathrm{d}\tau\\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big\{ -(\kappa+\mu_2)\big(\langle \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](v), u_\tau\rangle - \langle v, \widetilde{\mathrm{d}\alpha}^{\mu_1}[u](u_\tau)\rangle\big) - \langle \nabla H^{\mu_1}(u), v\rangle \Big\}\,\mathrm{d}\tau\\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle (\kappa+\mu_2)J^{\mu_1}(u)u_\tau - \nabla H^{\mu_1}(u), v\rangle\,\mathrm{d}\tau\\
&= (F(u, \mu_1, \mu_2), v)
\end{aligned}$$

for $v \in \mathcal{X}$, while the second is a consequence of the periodicity of u and hypothesis (H2). □

The next step is a Lyapunov-Schmidt reduction. Define

$$\begin{aligned}\mathcal{W}_1 &= \{u_0 + Ae_1 e^{i\tau} + Be_2 e^{i\tau} + \bar{A}\bar{e}_1 e^{-i\tau} + \bar{B}\bar{e}_2 e^{-i\tau},\ A, B \in \mathbb{C},\ u_0 \in Z\},\\ \mathcal{W}_2 &= \{u \in \mathcal{Z}\colon [u]_0 = 0,\ \Pi_{i\kappa}[u]_1 = \Pi_{-i\kappa}[u]_{-1} = 0\},\end{aligned}$$

where $\Pi_{\pm i\kappa}$ are the orthogonal projections onto the eigenspaces $E_{i\kappa} = \operatorname{span}\{e_1, e_2\}$ and $E_{-i\kappa} = \operatorname{span}\{\bar{e}_1, \bar{e}_2\}$, so that

$$\begin{aligned}\mathcal{Z} &= \mathcal{W}_1 \oplus \mathcal{W}_2,\\ \mathcal{X} &= (\mathcal{W}_1 \cap \mathcal{X}) \oplus (\mathcal{W}_2 \cap \mathcal{X})\end{aligned}$$

and the decompositions are orthogonal. Let $\widetilde{\Pi}_{\mathcal{W}_1}$ be the projection of $\mathcal{Z}$ onto $\mathcal{W}_1$ along $\mathcal{W}_2$, write $u \in \mathcal{U}$ as

$$u = \underbrace{\widetilde{\Pi}_{\mathcal{W}_1} u}_{=:\, u_{\mathcal{W}_1}} + \underbrace{(I - \widetilde{\Pi}_{\mathcal{W}_1}) u}_{=:\, u_{\mathcal{W}_2}}$$

and equation (2.2) as

$$\widetilde{\Pi}_{\mathcal{W}_1} F(u_{\mathcal{W}_1} + u_{\mathcal{W}_2}, \mu_1, \mu_2) = 0, \tag{2.3}$$

$$(I - \widetilde{\Pi}_{\mathcal{W}_1}) F(u_{\mathcal{W}_1} + u_{\mathcal{W}_2}, \mu_1, \mu_2) = 0. \tag{2.4}$$

To solve equation (2.4) (for $u_{\mathcal{W}_2}$ as a function of $u_{\mathcal{W}_1}$, μ_1 and μ_2) it is necessary to examine the solvability conditions for the equations

$$(\pm i\kappa I - L)u = J^0(0)^{-1} f \tag{2.5}$$

and

$$Lu = J^0(0)^{-1} f, \tag{2.6}$$

where f is a given function in Z. Normalise $e_1^{\mu_1}$, $e_2^{\mu_1}$ such that

$$\Omega^{\mu_1}|_0(e_1^{\mu_1}, \bar{e}_1^{\mu_1}) = \pm i, \quad \Omega^{\mu_1}|_0(e_2^{\mu_1}, \bar{e}_2^{\mu_1}) = \pm i, \quad \Omega^{\mu_1}|_0(e_1^{\mu_1}, e_2^{\mu_1}) = 0, \quad \Omega^{\mu_1}|_0(e_1^{\mu_1}, \bar{e}_2^{\mu_1}) = 0$$

and f_1, f_2 such that

$$\Omega^0|_0(f_1, f_2) = 1,$$

where $\Omega^{\mu_1}|_0$ is extended *bilinearly* to the complexification of Z. Observing that L is the Hamiltonian vector field for the linear Hamiltonian system $(Z, \Omega^0|_0, H_2^0)$, we find that the spectral projections $P_{\pm i\kappa}$ and P_0 onto the eigenspaces $E_{i\kappa} = \operatorname{span}\{e_1, e_2\}$, $E_{-i\kappa} = \operatorname{span}\{\bar{e}_1, \bar{e}_2\}$ and generalised eigenspace $E_0 = \operatorname{span}\{f_1, f_2\}$ are given by

$$\begin{aligned}P_{i\kappa} u &= \sum_{i=1}^{2} s_i \Omega^0|_0(u, \bar{e}_i) e_i = \sum_{i=1}^{2} s_i \langle J^0(0)(u), e_i \rangle e_i\\ P_{-i\kappa} u &= -\sum_{i=1}^{2} s_i \Omega^0|_0(u, e_i) \bar{e}_i = -\sum_{i=1}^{2} s_i \langle J^0(0)(u), \bar{e}_i \rangle \bar{e}_i,\end{aligned}$$

where $s_i = -\Omega^{\mu_1}|_0(e_i^{\mu_1}, \bar{e}_i^{\mu_1})$, and

$$\begin{aligned} P_0 u &= \Omega^0|_0(u, f_2) f_1 - \Omega^0|_0(u, f_1) f_2, \\ &= \langle J^0(0)(u), f_2\rangle f_1 - \langle J^0(0)(u), f_1\rangle f_2 \end{aligned}$$

(see Mielke [13, §3.1]); here $\langle\cdot\,,\cdot\rangle$ is extended *sesquilinearly* to the complexification of Z. This observation shows in particular that the (necessary and sufficient) solvability condition for (2.5), namely that the spectral projection of its right-hand side onto $E_{\pm \mathrm{i}\kappa}$ vanishes, is equivalent to the requirement that the orthogonal projection $\Pi_{\pm\mathrm{i}\kappa} f$ of f onto $E_{\pm\mathrm{i}\kappa}$ vanishes. In this case it has a unique solution in the orthogonal complement of $E_{\pm\mathrm{i}\kappa}$ in X which depends continuously upon f. Similarly, equation (2.6) is solvable if the orthogonal projection $\Pi_0 f$ of f onto E_0 vanishes (note that this is merely a sufficient condition), and in this case has a unique solution in the orthogonal complement of E_0 in X which depends continuously upon f. In the following analysis we use the convention that $f_1 = f_2 = 0$ and hence $P_0 = \Pi_0 = 0$ if 0 is not an eigenvalue of L.

Proposition 2.2. *The linear operator*

$$(I - \widetilde{\Pi}_{\mathcal{W}_1})\mathrm{d}_1 F[0,0,0] \colon (\mathcal{W}_2 \cap \mathcal{X}) \to \mathcal{W}_2$$

is an isomorphism.

Proof. The equation

$$(I - \widetilde{\Pi}_{\mathcal{W}_1})\mathrm{d}_1 F[0,0,0](v) = w \tag{2.7}$$

with $w \in \mathcal{W}_2$ is equivalent to

$$(\mathrm{i}\kappa k I - L)[v]_k = J^0(0)^{-1}[w]_k, \qquad k \in \mathbb{Z}\setminus\{0\},$$

with $[w]_k \in Z$, $k \notin \{0,-1,1\}$ and $[w]_1 \in E_{\mathrm{i}\kappa}^\perp$, $[w]_{-1} \in E_{-\mathrm{i}\kappa}^\perp$ (in X). By assumption (H5) the operator $\mathrm{i}\kappa k I - L \colon X \to Z$ is an isomorphism for $k \notin \{0,-1,1\}$ and we have established that the equations

$$\begin{aligned} (\mathrm{i}\kappa I - L)[v]_1 &= J^0(0)^{-1}[w]_1, \\ (-\mathrm{i}\kappa I - L)[v]_{-1} &= J^0(0)^{-1}[w]_{-1} \end{aligned}$$

have unique solutions $[v]_1 \in E_{\mathrm{i}\kappa}^\perp$, $[v]_{-1} \in E_{-\mathrm{i}\kappa}^\perp$ (in X) which depend continuously upon $[w]_1$, $[w]_{-1}$. It follows that

$$\begin{aligned} \|v\|^2_{L^2_{\mathrm{per}}(\mathbb{R},X)} &= \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \|[v]_k\|_X^2 + \|[v]_1\|_X^2 + \|[v]_{-1}\|_X^2 \\ &= \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \left\|(\mathrm{i}\kappa k I - L)^{-1} J^0(0)^{-1}[w]_k\right\|_X^2 + \|[v]_1\|_X^2 + \|[v]_{-1}\|_X^2 \\ &\lesssim \sum_{k\in\mathbb{Z}\setminus\{0,-1,1\}} \|[w]_k\|_Z^2 + \|[w]_1\|_Z^2 + \|[w]_{-1}\|_Z^2 \\ &\leq \|w\|_{\mathcal{Z}}^2 \end{aligned}$$

and similarly

$$\|v\|_{H^1_{\mathrm{per}}(\mathbb{R},Z)} \lesssim \|w\|_{\mathcal{Z}}$$

(by assumption (H6)), so that v lies in $\mathcal{X}$. We conclude that equation (2.7) has a unique solution $v \in \mathcal{W}_2 \cap \mathcal{X}$ which depends continuously upon $w \in \mathcal{W}_2$. □

Lemma 2.3. *There exist neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of the origin in respectively $\mathcal{W}_1$ and $\mathcal{W}_2$ and a reduction function $u_{\mathcal{W}_2} : \mathcal{U}_1 \times \Lambda_1 \times \Lambda_2 \to \mathcal{U}_2$ such that equation* (2.4) *admits the unique solution $(u_{\mathcal{W}_1}, u_{\mathcal{W}_2}(u_{\mathcal{W}_1}, \mu_1, \mu_2))$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Furthermore $u_{\mathcal{W}_1}(0,0,0) = 0$ and $\mathrm{d}u_{\mathcal{W}_1}[0,0,0] = 0$.*

Proof. This result follows from the implicit-function theorem and Proposition 2.2. □

The next step is to further decompose the reduced equation

$$\widetilde{\Pi}_{\mathcal{W}_1} F(u_{\mathcal{W}_1} + u_{\mathcal{W}_2}(u_{\mathcal{W}_1}, \mu_1, \mu_2), \mu_1, \mu_2) = 0 \tag{2.8}$$

by introducing the orthogonal projection $\widetilde{\Pi}$ of $\mathcal{Z}$ onto

$$\mathcal{W}_{1,1} = \{q f_1 + p f_2 + A e_1 \mathrm{e}^{\mathrm{i}\tau} + B e_2 \mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1 \mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2 \mathrm{e}^{-\mathrm{i}\tau},\ q, p \in \mathbb{R},\ A, B \in \mathbb{C}\},$$

which is given by

$$\widetilde{\Pi} u = \Pi_0 [u]_0 + \mathrm{e}^{\mathrm{i}\tau} \Pi_{\mathrm{i}\kappa}[u]_1 + \mathrm{e}^{-\mathrm{i}\tau}\Pi_{-\mathrm{i}\kappa}[u]_{-1},$$

so that

$$\mathcal{W}_1 = \mathcal{W}_{1,1} \oplus \mathcal{W}_{1,2},$$

where $\mathcal{W}_{1,2} = (I - \widetilde{\Pi})\mathcal{W}_1$. Writing $u_{\mathcal{W}_1} \in \mathcal{U}_1$ as

$$u_{\mathcal{W}_1} = \underbrace{\widetilde{\Pi} u_{\mathcal{W}_1}}_{=:\, u_1} + \underbrace{(I - \widetilde{\Pi}) u_{\mathcal{W}_1}}_{=:\, u_2},$$

we find that (2.8) is equivalent to

$$\widetilde{\Pi} G(u_1, u_2, \mu_1, \mu_2) = 0, \tag{2.9}$$

$$\underbrace{(I - \widetilde{\Pi}) G(u_1, u_2, \mu_1, \mu_2)}_{= (I - \Pi_0)[G(u_1, u_2, \mu_1, \mu_2)]_0} = 0, \tag{2.10}$$

where

$$G(u_1, u_2, \mu_1, \mu_2) = F(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2).$$

We proceed with a further reduction of Lyapunov-Schmidt type. Solving equation (2.10) for u_2 in terms of u_1, μ_1 and μ_2 requires hypothesis (H8) if the origin lies in the continuous spectrum of L or is an eigenvalue embedded in the continuous spectrum.

Lemma 2.4. *There exist neighbourhoods $\mathcal{U}_{1,1}$ and $\mathcal{U}_{1,2}$ of the origin in respectively $\mathcal{W}_{1,1}$ and $\mathcal{W}_{1,2}$ and a reduction function $u_2 : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to \mathcal{U}_{1,2}$ such that equation* (2.10) *admits the unique solution $(u_1, u_2(u_1, \mu_1, \mu_2))$ in $\mathcal{U}_{1,1} \times \mathcal{U}_{1,2}$. Furthermore $u_1(0,0,0) = 0$ and $\mathrm{d}u_1[0,0,0] = 0$.*

Proof. Equation (2.10) is equivalent to

$$L u_2 = J^0(0)^{-1}(I - \Pi_0) N^\star(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2),$$

Let $v(u_1, u_2, \mu_1, \mu_2)$ be the unique solution of the equation

$$Lv = J^0(0)^{-1}(I - \Pi_0)N^\star(u_1 + u_2 + u_{\mathcal{W}_2}(u_1 + u_2, \mu_1, \mu_2), \mu_1, \mu_2)$$

and define $\Upsilon : (\mathcal{U}_1 \cap \mathcal{W}_{1,1}) \times (\mathcal{U}_1 \cap \mathcal{W}_{1,2}) \times \Lambda_1 \times \Lambda_2 \to \mathcal{W}_{1,2}$ by

$$\Upsilon(u_1, u_2, \mu_1, \mu_2) = u_2 - v(u_1, u_2, \mu_1, \mu_2).$$

Observing that $\Upsilon(0,0,0,0) = 0$, $\mathrm{d}_2\Upsilon[0,0,0,0] = I$, one therefore obtains the result from the implicit-function theorem. □

The reduced equation

$$\widetilde{\Pi} G(u_1, u_2(u_1, \mu_1, \mu_2), \mu_1, \mu_2) = 0$$

is conveniently written as

$$f(u_1, \mu_1, \mu_2) = 0, \tag{2.11}$$

where

$$f(u_1, \mu_1, \mu_2) = \widetilde{\Pi} F(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2)$$

and the new reduction function $h : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to (I - \widetilde{\Pi})\mathcal{X}$ is given by

$$h(u_1, \mu_1, \mu_2) = u_2(u_1, \mu_1, \mu_2) + u_{\mathcal{W}_2}(u_1 + u_2(u_1, \mu_1, \mu_2), \mu_1, \mu_2).$$

Note again that $h(0,0,0) = 0$ and $\mathrm{d}_1 h[0,0,0] = 0$.

Equation (2.11) inherits the variational structure of (2.2).

Proposition 2.5. *Equation* (2.11) *is the Euler-Lagrange equation for the reduced action functional $s : \mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$ given by*

$$s(u_1, \mu_1, \mu_2) = S(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2),$$

that is

$$\mathrm{d}_1 s[u_1, \mu_1, \mu_2](v_1) = (f(u_1, \mu_1, \mu_2), v_1) \tag{2.12}$$

for all $v_1 \in \widetilde{\Pi}\mathcal{X}$.

Proof. This result follows from the calculation

$$\begin{aligned}
\mathrm{d}_1 s[u_1, \mu_1, \mu_2](v_1) &= \mathrm{d}_1 S[u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2](v_1 + \mathrm{d}_1 h[u_1, \mu_1, \mu_2](v_1)) \\
&= (F(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2), \mu_1, \mu_2), \mathrm{d}_1 h[u_1, \mu_1, \mu_2](v_1) + v_1) \\
&= (\widetilde{\Pi} F(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2), \mathrm{d}_1 h[u_1, \mu_1, \mu_2](v_1) + v_1) \\
&= (\widetilde{\Pi} F(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2), v_1) \\
&= (f(u_1, \mu_1, \mu_2), v_1),
\end{aligned}$$

where the second line follows from the first by Proposition 2.1, the third follows from the second because

$$(I - \widetilde{\Pi})F(u_1 + h(u_1, \mu_1, \mu_2), \mu_1, \mu_2) = 0$$

by construction, and the fourth follows from the third because $h(u_1, \mu_1, \mu_2)$ (and hence all its derivatives) lies in $(I - \widetilde{\Pi})\tilde{X}$. □

Introducing coordinates

$$u_1 = qf_1 + pf_2 + Ae_1\mathrm{e}^{\mathrm{i}\tau} + Be_2\mathrm{e}^{\mathrm{i}\tau} + \bar{A}\bar{e}_1\mathrm{e}^{-\mathrm{i}\tau} + \bar{B}\bar{e}_2\mathrm{e}^{-\mathrm{i}\tau},$$

one finds that the reduced equation (2.11) is given by

$$\partial_{\bar{A}} s = 0, \tag{2.13}$$
$$\partial_{\bar{B}} s = 0, \tag{2.14}$$
$$\partial_p s = 0 \tag{2.15}$$

(recall that s does not depend upon q by hypothesis (H7)). The reduced action functional s remains invariant under the symmetries T_θ and T, whose actions on $\mathcal{W}_{1,1}$ are given by

$$\begin{aligned} T_\theta(A, B, \bar{A}, \bar{B}, q, p) &= (A\mathrm{e}^{\mathrm{i}\theta}, B\mathrm{e}^{\mathrm{i}\theta}, \bar{A}\mathrm{e}^{-\mathrm{i}\theta}, \bar{B}\mathrm{e}^{-\mathrm{i}\theta}, q, p), \\ T(A, B, \bar{A}, \bar{B}, q, p) &= (\bar{A}, \bar{B}, A, B, -q, p). \end{aligned}$$

It follows that s is a real-valued function of the real quantities $|A|^2$, $|B|^2$, $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B})$, $\frac{1}{2}(A\bar{B} + \bar{A}B)$, p, μ_1 and μ_2 which is even with respect to $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B})$. Restricting to $A = r_1$, $B = \mathrm{i}r_2$, where r_1 and r_2 are real (so that $\frac{\mathrm{i}}{2}(\bar{A}B - A\bar{B}) = r_1r_2$, $\frac{1}{2}(A\bar{B} + \bar{A}B) = 0$), we find that

$$s(A, B, \bar{A}, \bar{B}, p, \mu_1, \mu_2) = \tilde{s}(r_1^2, r_2^2, r_1r_2, p, \mu_1, \mu_2),$$

where the right-hand side is even in its third argument, so that in fact, with a slight abuse of notation,

$$s(A, B, \bar{A}, \bar{B}, q, \mu_1, \mu_2) = \tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2).$$

Equations (2.13)–(2.15) therefore reduce to

$$\begin{aligned} r_1\partial_1\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) &= 0, \\ r_2\partial_2\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) &= 0, \\ \partial_3\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) &= 0, \end{aligned}$$

and further to

$$\partial_1\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0, \tag{2.16}$$
$$\partial_2\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0, \tag{2.17}$$
$$\partial_3\tilde{s}(r_1^2, r_2^2, p, \mu_1, \mu_2) = 0 \tag{2.18}$$

for solutions with non-zero r_1 and r_2 components.

Lemma 2.6. *The quadratic parts of $\tilde{s}$ which are respectively independent of (μ_1, μ_2), independent of μ_2 and linear in μ_1, and independent of μ_1 and linear in μ_2 are given by*

$$\begin{aligned} \tilde{s}_2^{00} &= \tilde{s}_{002}^{00}p^2, \\ \tilde{s}_2^{10} &= \tilde{s}_{200}^{10}\mu_1r_1^2 + \tilde{s}_{020}^{10}\mu_1r_2^2 + \tilde{s}_{002}^{10}\mu_1p^2, \\ \tilde{s}_2^{01} &= -s_1\mu_2r_1^2 - s_2\mu_2r_2^2, \end{aligned}$$

where

$$\begin{aligned}
\tilde{s}^{00}_{002} &= -\tfrac{1}{2},\\
\tilde{s}^{10}_{200} &= \Omega^0_0(\partial_{\mu_1} L^0 e_1, \bar{e}_1),\\
\tilde{s}^{10}_{020} &= \Omega^0_0(\partial_{\mu_1} L^0 e_2, \bar{e}_2),\\
\tilde{s}^{10}_{002} &= \tfrac{1}{2}\Omega^0_0(\partial_{\mu_1} L^0 f_2, f_2).
\end{aligned}$$

Proof. We begin by recording the formulae

$$\begin{aligned}
\mathrm{d}_1^2 S[0,0,0](v_1,v_2) &= (J^0(0)(\kappa v_{1\tau} - L v_1), v_2),\\
\mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](v_1,v_2,1) &= (\partial_{\mu_1} J^0(0)(\kappa v_{1\tau} - L v_1) + J^0(0)\partial_{\mu_1} L^0 v_1, v_2),\\
\mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](v_1,v_2,1) &= (J^0(0) v_{1\tau}, v_2)
\end{aligned}$$

for v_1, $v_2 \in \mathcal{X}$, which are obtained by differentiating the identity

$$\mathrm{d}_1 S[u,\mu_1,\mu_2](v) = \big(J^{\mu_1}(u)\big((\kappa+\mu_2)u_\tau - L^{\mu_1}u - N^{\mu_1}(u)\big), v\big)$$

for $(u,\mu_1,\mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$ and $v \in \mathcal{X}$ (see Proposition 2.1).

These formulae show that

$$\tilde{s}^{00}_2 = \tilde{s}^{00}_{200} r_1^2 + \tilde{s}^{00}_{020} r_2^2 + \tilde{s}^{00}_{002} p^2,$$

where

$$\begin{aligned}
\tilde{s}^{00}_{200} &= \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1) = (J^0(0)(\mathrm{i}\kappa I - L)\mathrm{e}^{\mathrm{i}\tau} e_1, \mathrm{e}^{\mathrm{i}\tau} e_1) = 0,\\
\tilde{s}^{00}_{020} &= \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2) = (J^0(0)(\mathrm{i}\kappa I - L)\mathrm{e}^{\mathrm{i}\tau} e_2, \mathrm{e}^{\mathrm{i}\tau} e_2) = 0,\\
\tilde{s}^{00}_{002} &= \tfrac{1}{2}\mathrm{d}_1^2 S[0,0,0](f_2,f_2) = -\tfrac{1}{2}(J^0(0)Lf_2, f_2) = -\tfrac{1}{2}\Omega^0_0(f_1,f_2) = -\tfrac{1}{2}.
\end{aligned}$$

Similarly, denoting the part of $h(u_1,\mu_1,\mu_2)$ which is homogeneous of degree i, j, k, ℓ, n_1 and n_2 in respectively A, B, $\bar{A}$, $\bar{B}$, μ_1 and μ_2 by $h^{n_1 n_2}_{ijk\ell} A^i B^j \bar{A}^k \bar{B}^\ell \mu_1^{n_1}\mu_2^{n_2}$, we find that

$$\begin{aligned}
\tilde{s}^{10}_2 &= \tilde{s}^{10}_{200}\mu_1 r_1^2 + \tilde{s}^{10}_{020}\mu_1 r_2^2 + \tilde{s}^{10}_{002}\mu_1 p^2,\\
\tilde{s}^{01}_2 &= \tilde{s}^{01}_{200}\mu_2 r_1^2 + \tilde{s}^{01}_{020}\mu_2 r_2^2 + \tilde{s}^{01}_{002}\mu_2 p^2,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{s}^{10}_{200} &= \mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_1, h^{10}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{10}_{10000}),\\
&= (J^0(0)\partial_{\mu_1} L^0(\mathrm{e}^{\mathrm{i}\tau} e_1), \mathrm{e}^{\mathrm{i}\tau} e_1)\\
&= \Omega^0_0(\partial_{\mu_1} L^0(e_1), \bar{e}_1),\\
\tilde{s}^{10}_{020} &= \mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau} e_2, h^{10}_{00010}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{10}_{01000})\\
&= (J^0(0)\partial_{\mu_1} L^0(\mathrm{e}^{\mathrm{i}\tau} e_2), \mathrm{e}^{\mathrm{i}\tau} e_2)\\
&= \Omega^0_0(\partial_{\mu_1} L^0(e_2), \bar{e}_2),\\
\tilde{s}^{10}_{002} &= \tfrac{1}{2}\mathrm{d}_1^2\mathrm{d}_2 S[0,0,0](f_2,f_2,1) + \mathrm{d}_1^2 S[0,0,0](f_2, h^{10}_{00001})
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}(J^0(0)\partial_{\mu_1}L^0 f_2, f_2)\\
&= \tfrac{1}{2}\Omega_0^0(\partial_{\mu_1}L^0 f_2, f_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{s}^{01}_{200} &= \mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_1, h^{01}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_1, h^{01}_{10000})\\
&= \mathrm{i}(J^0(0)\mathrm{e}^{\mathrm{i}\tau}e_1, \mathrm{e}^{\mathrm{i}\tau}e_1)\\
&= \mathrm{i}\Omega_0^0(e_1, \bar{e}_1)\\
&= -s_1,\\
\tilde{s}^{01}_{020} &= \mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, \mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, 1) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{\mathrm{i}\tau}e_2, h^{01}_{00100}) + \mathrm{d}_1^2 S[0,0,0](\mathrm{e}^{-\mathrm{i}\tau}\bar{e}_2, h^{01}_{01000})\\
&= \mathrm{i}(J^0(0)\mathrm{e}^{\mathrm{i}\tau}e_2, \mathrm{e}^{\mathrm{i}\tau}e_2)\\
&= \mathrm{i}\Omega_0^0(e_2, \bar{e}_2)\\
&= -s_2,\\
\tilde{s}^{01}_{002} &= \tfrac{1}{2}\mathrm{d}_1^2\mathrm{d}_3 S[0,0,0](f_2, f_2, 1) + \mathrm{d}_1^2 S[0,0,0](f_2, h^{10}_{00001})\\
&= 0.
\end{aligned}
$$

Note that the second derivatives are extended *bilinearly* to the complexification of $\mathcal{Z}$ while $(\cdot\,,\cdot)$ is extended *sesquilinearly*. □

Corollary 2.7. *One has the formulae*

$$
\tilde{s}^{10}_{200} = \partial_{\mu_1}\kappa_1^0, \qquad \tilde{s}^{01}_{200} = \partial_{\mu_1}\kappa_2^0.
$$

Proof. Observe that

$$
\Omega_0^{\mu_1}(L^{\mu_1}e_i^{\mu_1}, \bar{e}_i^{\mu_1}) = \mathrm{i}\kappa_i^{\mu_1}\Omega_0^{\mu_1}(e_i^{\mu_1}, \bar{e}_i^{\mu_1}) = -s_i\kappa_i^{\mu_1}.
$$

Differentiating this formula with respect to μ_1 and evaluating the result at $\mu_1 = 0$ yields

$$
\begin{aligned}
-s_i\partial_{\mu_1}\kappa_i^0\big|_{\mu_1=0} &= \Omega_0^0(\partial_{\mu_1}L^0 e_i, \bar{e}_i) + \Omega_0^1(Le_i, \bar{e}_i) + \Omega_0^0(L\partial_{\mu_1}e_i^{\mu_1}, \bar{e}_i) + \Omega_0^0(Le_i, \partial_{\mu_1}\bar{e}_i^{\mu_1})\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i, \bar{e}_i) + \Omega_0^1(Le_i, \bar{e}_i) - \Omega_0^0(\partial_{\mu_1}e_i^{\mu_1}, L\bar{e}_i) + \Omega_0^0(Le_i, \partial_{\mu_1}\bar{e}_i^{\mu_1})\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i, \bar{e}_i) + \mathrm{i}\kappa\big(\Omega_0^1(e_i, \bar{e}_i) + \Omega_0^0(\partial_{\mu_1}e_i^{\mu_1}, \bar{e}_i) + \Omega_0^0(e_i, \partial_{\mu_1}\bar{e}_i^{\mu_1})\big)\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i, \bar{e}_i) + \mathrm{i}\kappa\partial_{\mu_1}\underbrace{\Omega_0^{\mu_1}(e_i^{\mu_1}, \partial_{\mu_1}\bar{e}_i^{\mu_1})}_{=\,s_i}\Big|_{\mu_1=0}\\
&= \Omega_0^0(\partial_{\mu_1}L^0 e_i, \bar{e}_i).
\end{aligned}
$$

□

Finally, we solve equations (2.16)–(2.18) using the information given by Lemma 2.6 and Corollary 2.7, thus completing the proof of Theorem 1.2.

Lemma 2.8. *There exist* $\varepsilon > 0$ *and functions* $p^\star : B_\varepsilon(0) \to \mathbb{R}$, $\mu_1^\star : B_\varepsilon(0) \to \mathbb{R}$, $\mu_2^\star : B_\varepsilon(0) \to \mathbb{R}$ *such that the solution set of* (2.16)–(2.18) *in* $\mathcal{U}_{1,1} \times \Lambda_1 \times \Lambda_2$ *coincides with*

$$\{(r_1^2, r_2^2, p^\star(r_1^2, r_2^2), \mu_1^\star(r_1^2, r_2^2), \mu_2^\star(r_1^2, r_2^2) : |(r_1^2, r_2^2)| < \varepsilon\}.$$

Proof. Since

$$\begin{pmatrix} \partial_1 \tilde{s}(0,0,0,0,0) \\ \partial_2 \tilde{s}(0,0,0,0,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} \det \begin{pmatrix} \partial_1\partial_4 \tilde{s}(0,0,0,0,0) & \partial_1\partial_5 \tilde{s}(0,0,0,0,0) \\ \partial_2\partial_4 \tilde{s}(0,0,0,0,0) & \partial_2\partial_5 \tilde{s}(0,0,0,0,0) \end{pmatrix} &= -s_2 \tilde{s}_{200}^{10} + s_1 \tilde{s}_{020}^{01} \\ &= -s_1 s_2 \partial_{\mu_1}(\kappa_1^0 - \kappa_2^0) \\ &\neq 0, \end{aligned}$$

we can solve equations (2.16), (2.17) locally for $\mu_1 = \mu_1(r_1^2, r_2^2, p)$, $\mu_2 = \mu_2(r_1^2, r_2^2, p)$ using the implicit-function theorem. Inserting this solution into (2.18) yields

$$\tilde{t}(r_1^2, r_2^2, p) = 0, \qquad (2.19)$$

where

$$\tilde{t}(r_1^2, r_2^2, p) = \partial_3 \tilde{s}(r_1^2, r_2^2, p, \mu_1(r_1^2, r_2^2, p), \mu_2(r_1^2, r_2^2, p)).$$

Furthermore

$$\tilde{t}(0,0,0) = \partial_3 \tilde{s}(0,0,0, \mu_1(0,0,0), \mu_2(0,0,0)) = 0$$

and similarly

$$\begin{aligned} \partial_3 \tilde{t}(0,0,0) &= \partial_3^2 \tilde{s}(0,0,0,0,0) + \partial_3\mu_1(0,0,0)\partial_3\partial_4 \tilde{s}(0,0,0,0,0) + \partial_3\mu_2(0,0,0)\partial_3\partial_5 \tilde{s}(0,0,0,0,0) \\ &= 2\tilde{s}_{002}^{00} \\ &= -1. \end{aligned}$$

We can therefore solve equation (2.18) locally for $p = p(r_1^2, r_2^2)$ using the implicit-function theorem.

The assertion follows by setting $p^\star(r_1^2, r_2^2) = p(r_1^2, r_2^2)$, $\mu_1^\star(r_1^2, r_2^2) = \mu_1(r_1^2, r_2^2, p(r_1^2, r_2^2))$ and $\mu_2^\star(r_1^2, r_2^2) = \mu_2(r_1^2, r_2^2, p(r_1^2, r_2^2))$. □

3 Hydroelastic waves

In this section we introduce the hydrodynamic problem for travelling waves on the surface of a three-dimensional body of water beneath a thin ice sheet modelled using the Cosserat theory of hyperelastic shells (Plotnikov & Toland [14]). The fluid is bounded below by a rigid horizontal bottom $\{x_2 = -h\}$ (the cases $h < \infty$ and $h = \infty$ are referred to as 'finite depth' and 'infinite depth') and above by a free surface $\{x_2 = \eta(x_1, x_3)\}$ (in a frame of reference following the wave with constant speed c in the x_1 direction); there is no cavitation between this surface and the ice sheet. Working in a dimensionless coordinates with unit length $(D/\rho g)^{1/4}$ and unit speed $(\rho/Dg^3)^{1/8}$, one finds that the hydrodynamic problem is to find an Eulerian velocity potential ϕ which satisfies the equations

$$\phi_{x_1x_1} + \phi_{x_2x_2} + \phi_{x_3x_3} = 0, \qquad -\tfrac{1}{\beta} < x_2 < \eta(x_1, x_3), \tag{3.1}$$

$$\phi_{x_2} = 0, \qquad x_2 = -\tfrac{1}{\beta}, \tag{3.2}$$

$$\phi_{x_2} + \gamma\eta_{x_1} - \phi_{x_1}\eta_{x_1} - \phi_{x_3}\eta_{x_3} = 0, \qquad x_2 = \eta(x_1, x_3), \tag{3.3}$$

$$-\gamma\phi_{x_1} + \tfrac{1}{2}(\phi_{x_1}^2 + \phi_{x_2}^2 + \phi_{x_3}^2) + \eta + U(\eta) = 0, \qquad x_2 = \eta(x_1, x_3), \tag{3.4}$$

where

$$\begin{aligned} U(\eta) = 2\Bigg(\frac{1}{\sqrt{Q(\eta)}}\Bigg[\partial_{x_1}\left(\frac{1+\eta_{x_3}^2}{\sqrt{Q(\eta)}}P(\eta)_{x_1}\right) - \partial_{x_1}\left(\frac{\eta_{x_1}\eta_{x_3}}{\sqrt{Q(\eta)}}P(\eta)_{x_3}\right) \\ - \partial_{x_3}\left(\frac{\eta_{x_1}\eta_{x_3}}{\sqrt{Q(\eta)}}P(\eta)_{x_1}\right) + \partial_{x_3}\left(\frac{1+\eta_{x_1}^2}{\sqrt{Q(\eta)}}P(\eta)_{x_3}\right)\Bigg] + 2P(\eta)^3 - 2K(\eta)P(\eta)\Bigg), \end{aligned}$$

$$Q(\eta) = 1 + \eta_{x_1}^2 + \eta_{x_3}^2,$$

$$P(\eta) = \frac{1}{2Q(\eta)^{3/2}}\Big[(1+\eta_{x_3}^2)\eta_{x_1x_1} - 2\eta_{x_1x_3}\eta_{x_1}\eta_{x_3} + (1+\eta_{x_1}^2)\eta_{x_3x_3}\Big],$$

$$K(\eta) = \frac{1}{Q(\eta)^2}(\eta_{x_1x_1}\eta_{x_3x_3} - \eta_{x_1x_3}^2),$$

and

$$\beta = \left(\frac{D}{\rho g h^4}\right)^{1/4} \geq 0, \qquad \gamma = \left(\frac{c^8\rho}{Dg^3}\right)^{1/8} > 0,$$

where D, ρ and g are respectively the coefficient of flexural rigidity for the ice sheet, the density of the fluid and the acceleration due to gravity (see Guyenne and Parau [6]).

We consider waves which are periodic with periods p_1 and p_2 in two arbitrary horizontal directions x and z which form (different) angles θ_1, $\theta_2 \in [0, \pi)$ with the x_1-axis respectively, so that

$$x = \csc(\theta_2 - \theta_1)(x_1 \sin\theta_2 - x_3\cos\theta_2), \qquad z = \csc(\theta_1 - \theta_2)(x_1\sin\theta_1 - x_3\cos\theta_1)$$

(see Figure 1). To this end we seek solutions of the governing equations of the form

$$\eta(x_1, x_3) = \tilde{\eta}(\tilde{x}, \tilde{z}), \qquad \phi(x_1, x_2, x_3) = \tilde{\phi}(\tilde{x}, x_2, \tilde{z}),$$

where

$$\tilde{x} = x_1\sin\theta_2 - x_3\cos\theta_2, \qquad \tilde{z} = \frac{2\pi}{p_2}(x_1\sin\theta_1 - x_3\cos\theta_1)$$

and $\tilde{\eta}$, $\tilde{\phi}$ are 2π-periodic in $\tilde{z}$ (the requirement that they are also periodic in $\tilde{x}$ is applied later). The governing equations become

$$\phi_{xx} + \phi_{x_2x_2} + \nu^2\phi_{zz} + 2\nu\cos(\theta_1 - \theta_2)\phi_{xz} = 0, \qquad -\tfrac{1}{\beta} < x_2 < \eta, \tag{3.5}$$

$$\phi_{x_2} = 0, \qquad x_2 = -\tfrac{1}{\beta}, \tag{3.6}$$

$$\begin{aligned}\phi_{x_2} = &-\gamma(\sin\theta_2\eta_x + \nu\sin\theta_1\eta_z) + \eta_x\phi_x + \nu^2\eta_z\phi_z \\ &+ \nu\cos(\theta_1-\theta_2)(\eta_x\phi_z + \eta_z\phi_x), \qquad x_2 = \eta,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&-\gamma(\sin\theta_2\phi_x + \nu\sin\theta_1\phi_z) \\ &\quad + \tfrac{1}{2}(\phi_x^2 + \phi_{x_2}^2 + \nu^2\phi_z^2 + 2\nu\cos(\theta_1-\theta_2)\phi_x\phi_z) + \gamma\eta + U(\eta) = 0, \qquad x_2 = \eta,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}U(\eta) = 2\Bigg(&\frac{1}{\sqrt{Q(\eta)}}\Bigg[\partial_x\Bigg(\frac{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}{\sqrt{Q(\eta)}}P(\eta)_x\Bigg) + \nu^2\partial_z\Bigg(\frac{1+\sin^2(\theta_1-\theta_2)\eta_x^2}{\sqrt{Q(\eta)}}P(\eta)_z\Bigg) \\ &+ \nu\cos(\theta_1-\theta_2)\Bigg(\partial_x\Bigg(\frac{P(\eta)_z}{\sqrt{Q(\eta)}}\Bigg) + \partial_z\Bigg(\frac{P(\eta)_x}{\sqrt{Q}}\Bigg)\Bigg) \\ &- \nu^2\sin^2(\theta_1-\theta_2)\Bigg(\partial_x\Bigg(\frac{\eta_x\eta_z}{\sqrt{Q(\eta)}}P(\eta)_z\Bigg) + \partial_z\Bigg(\frac{\eta_x\eta_z}{\sqrt{Q(\eta)}}P(\eta)_x\Bigg)\Bigg]\Bigg. \\ &+ 2P(\eta)^3 - 2K(\eta)P(\eta)\Bigg),\end{aligned}$$

$$Q(\eta) = 1 + \eta_x^2 + \nu^2\eta_z^2 + 2\nu\cos(\theta_1-\theta_2)\eta_x\eta_z,$$

$$P(\eta) = \frac{1}{2Q(\eta)^{\frac{3}{2}}}\Big[\eta_{xx} + \nu^2\eta_{zz} + 2\nu\cos(\theta_1-\theta_2)\eta_{xz} + \nu^2\sin^2(\theta_1-\theta_2)(\eta_{xx}\eta_z^2 - 2\eta_{xz}\eta_x\eta_z + \eta_{zz}\eta_x^2)\Big],$$

$$K(\eta) = \frac{1}{Q(\eta)^2}\nu^2\sin^2(\theta_1-\theta_2)(\eta_{xx}\eta_{zz} - \eta_{xz}^2),$$

the tildes have been dropped for notational simplicity, and $\nu = 2\pi/p_2$.

We proceed by formulating equations (3.5)–(3.8) as a Hamiltonian system in which the horizontal spatial direction x plays the role of the time-like variable ('spatial dynamics'). Our starting point is the observation that these equations follow from the formal variational principle

$$\begin{aligned}\delta\int\int_0^{2\pi}\Bigg\{&\int_{-\frac{1}{\beta}}^{\eta}\frac{1}{2}\left(\phi_x^2 + \phi_{x_2}^2 + \nu^2\phi_z^2 + 2\nu\cos(\theta_1-\theta_2)\phi_x\phi_z\right)\mathrm{d}x_2 \\ &+ 2\sqrt{Q(\eta)}P(\eta)^2 + \tfrac{1}{2}\eta^2 + \gamma(\eta_x\sin\theta_2 + \nu\eta_z\sin\theta_1)\phi|_{x_2=\eta}\Bigg\}\,\mathrm{d}z\,\mathrm{d}x = 0,\end{aligned} \tag{3.9}$$

in which the variations are taken over η and ϕ (a modified version of the classical variational principle introduced by Luke [12]). Because of the difficulty in performing analysis on a variable domain, we use the change of variable

$$\phi(x, x_2, z) = \Phi(x, y, z), \qquad x_2 = \begin{cases} y + (1+\beta y)\eta, & \beta > 0, \\ y + \mathrm{e}^y\eta, & \beta = 0, \end{cases}$$

to map the variable fluid domain $\{-\frac{1}{\beta} < x_2 < \eta(x,z)\}$ to the fixed domain $\{-\frac{1}{\beta} < y < 0\}$. The variational principle (3.9) is transformed into

$$\delta\mathcal{L} = 0, \qquad \mathcal{L} = \int L(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x)\,\mathrm{d}x,$$

in which

$$
\begin{aligned}
&L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x)\\
&\quad=\int_0^{2\pi}\bigg\{\int_{-\frac{1}{\beta}}^{0}\frac{1}{2K_2(\eta)}\Big((\Phi_x-K_1(\eta)\eta_x\Phi_y)^2+K_2(\eta)^2\Phi_y^2+\nu^2(\Phi_z-K_1(\eta)\eta_z\Phi_y)\\
&\qquad\qquad+2\nu\cos(\theta_1-\theta_2)(\Phi_x-K_1(\eta)\eta_x\Phi_y)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\Big)\,\mathrm{d}y\\
&\qquad+2\sqrt{Q(\eta)}P(\eta)^2+\tfrac{1}{2}\eta^2+\gamma(\eta_x\sin\theta_2+\nu\eta_z\sin\theta_1)\Phi|_{y=0}\bigg\}\,\mathrm{d}z
\end{aligned}
$$

and

$$
K_1(\eta)=\begin{cases}\dfrac{1+\beta y}{1+\beta\eta}, & \beta>0,\\[2mm] \dfrac{\mathrm{e}^y}{1+\mathrm{e}^y\eta}, & \beta=0,\end{cases}\qquad K_2(\eta)=\begin{cases}\dfrac{1}{1+\beta\eta}, & \beta>0,\\[2mm] \dfrac{1}{1+\mathrm{e}^y\eta}, & \beta=0.\end{cases}
$$

The next step is to perform a formal Legendre transformation (see Lanczos [11, Appendix I]) by introducing the new coordinate

$$
\rho=\eta_x
$$

and momenta

$$
\begin{aligned}
\zeta&=\frac{\delta L}{\delta\eta_x}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta L}{\delta\eta_{xx}}\right)\\
&=-\int_{-\frac{1}{\beta}}^{0}\frac{K_1(\eta)}{K_2(\eta)}\Big((\Phi_x-K_1(\eta)\eta_x\Phi_y)\Phi_y+\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\Phi_y\Big)\,\mathrm{d}y\\
&\quad-10\frac{P(\eta)^2}{\sqrt{Q(\eta)}}(\eta_x+\nu\cos(\theta_1-\theta_2)\eta_z)\\
&\quad+\frac{4P(\eta)}{Q(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(-\eta_{xz}\eta_z+\eta_{zz}\eta_x)+\gamma\sin\theta_2\Phi|_{y=0}\\
&\quad-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2P(\eta)}{Q(\eta)}(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)\right),\\
\xi&=\frac{\delta L}{\delta\eta_{xx}}\\
&=\frac{2P(\eta)}{Q(\eta)}(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2),\\
\Psi&=\frac{\delta L}{\delta\Phi_x}\\
&=\frac{1}{K_2(\eta)}(\Phi_x-K_1(\eta)\eta_x\Phi_y)+\frac{1}{K_2(\eta)}\nu\cos(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y),
\end{aligned}
$$

and defining the Hamiltonian by

$H(\eta,\rho,\Phi,\zeta,\xi,\Psi)$

$$
\begin{aligned}
&= \int_S \zeta\eta_x \,\mathrm{d}z + \int_S \xi\eta_{xx}\,\mathrm{d}z + \int_\Sigma \Psi\Phi_x \,\mathrm{d}y\,\mathrm{d}z - L(\eta,\eta_x,\eta_{xx},\Phi,\Phi_x) \\
&= \int_\Sigma \Bigg\{ \frac{K_2(\eta)}{2}(\Psi^2 - \Phi_y^2) - \frac{1}{2K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)^2 \\
&\qquad + K_1(\eta)\rho\Phi_y\Psi - \nu\cos(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\Psi \Bigg\}\,\mathrm{d}y\,\mathrm{d}z \\
&\quad + \int_S \Bigg\{ \zeta\rho - \frac{1}{2}\eta^2 - \gamma(\rho\sin\theta_2 + \nu\eta_z\sin\theta_1)\Phi|_{y=0} + \frac{Q(\eta,\rho)^{5/2}\xi^2}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2} \\
&\qquad + \frac{\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\big(-(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz} \\
&\qquad\qquad + 2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z - 2\nu\cos(\theta_1-\theta_2)\rho_z\big)\Bigg\}\,\mathrm{d}z,
\end{aligned}
\tag{3.10}
$$

where $S=(0,2\pi)$, $\Sigma=(-\frac{1}{\beta},0)\times(0,2\pi)$ and

$$
Q(\eta,\rho) = 1+\rho^2+\nu^2\eta_z^2 + 2\nu\cos(\theta_1-\theta_2)\rho\eta_z.
$$

Hamilton's equations are

$$
\begin{aligned}
\eta_x &= \frac{\delta H}{\delta\zeta} \\
&= \rho,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\rho_x &= \frac{\delta H}{\delta\xi} \\
&= \frac{Q(\eta,\rho)^{5/2}\xi}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2} - \frac{(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz}}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2} \\
&\quad - \frac{2\nu\cos(\theta_1-\theta_2)\rho_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2} + \frac{2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\Phi_x &= \frac{\delta H}{\delta\Psi} \\
&= K_2(\eta)\Psi + K_1(\eta)\rho\Phi_y - \nu\cos(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
-\zeta_x &= \frac{\delta H}{\delta\eta} \\
&= \int_{-\frac{1}{\beta}}^0 \Bigg\{ -\frac{K_2(\eta)^2K_3}{2}(\Psi^2-\Phi_y^2) - \frac{K_3}{2}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)^2 \\
&\qquad - K_1(\eta)K_2(\eta)K_3\rho\Phi_y\Psi - \nu^2\sin^2(\theta_1-\theta_2)K_1(\eta)K_3\eta_z\Phi_y(\Phi_z - K_1(\eta)\eta_z\Phi_y) \\
&\qquad - \nu\cos(\theta_1-\theta_2)K_1(\eta)K_2(\eta)K_3\eta_z\Phi_y\Psi - [\nu\cos(\theta_1-\theta_2)K_1(\eta)\Phi_y\Psi]_z \\
&\qquad - \left[\frac{K_1(\eta)}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\Phi_y\right]_z \Bigg\}\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)Q(\eta,\rho)^{5/2}\xi^2\eta_z}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^3}\right]_z \\
&-\left[\frac{5\nu^2 Q(\eta,\rho)^{3/2}\xi^2\eta_z}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\right]_z-\left[\frac{5\nu\cos(\theta_1-\theta_2)Q(\eta,\rho)^{3/2}\xi^2\rho}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\right]_z \\
&+\Bigg[\frac{2\nu\sin^2(\theta_1-\theta_2)\xi\eta_z}{(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}\big(-(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\eta_{zz} \\
&\qquad\qquad +2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\eta_z-2\nu\cos(\theta_1-\theta_2)\rho_z\big)\Bigg]_z \\
&-\left[\frac{(1+\sin^2(\theta_1-\theta_2)\rho^2)\nu^2\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_{zz}-\left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)\rho\rho_z\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z-\eta+\gamma\nu\sin\theta_1\Phi_z|_{y=0},
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
-\xi_x &= \frac{\delta H}{\delta\rho} \\
&= \zeta-\gamma\sin\theta_2\Phi|_{y=0}+\int_{-\frac{1}{\beta}}^0 K_1(\eta)\Psi\Phi_y\,\mathrm{d}y+\frac{5Q(\eta,\rho)^{3/2}\xi^2\rho}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2} \\
&\quad+\frac{5\nu\cos(\theta_1-\theta_2)Q(\eta,\rho)^{3/2}\xi^2\eta_z}{2(1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2)^2}-\frac{2\nu^2\sin^2(\theta_1-\theta_2)\xi\rho\eta_{zz}}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2} \\
&\quad-\left[\frac{2\nu^2\sin^2(\theta_1-\theta_2)\xi\eta_z}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z\rho+\left[\frac{2\nu\cos(\theta_1-\theta_2)\xi}{1+\nu^2\sin^2(\theta_1-\theta_2)\eta_z^2}\right]_z,
\end{aligned} \tag{3.15}
$$

$$
-\Psi_x = \frac{\delta H}{\delta\Phi} \tag{3.16}
$$

$$
\begin{aligned}
&= (K_2(\eta)\Phi_y)_y-(K_1(\eta)\Psi\rho)_y-\left(\frac{K_1(\eta)}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\eta_z\right)_y \\
&\quad+\left[\frac{1}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\right]_z \\
&\quad+\nu\cos(\theta_1-\theta_2)\Psi_z-\nu\cos(\theta_1-\theta_2)(K_1(\eta)\eta_z\Psi)_y,
\end{aligned} \tag{3.17}
$$

where

$$
K_3=\begin{cases}\beta, & \beta>0,\\ \mathrm{e}^y, & \beta=0,\end{cases}
$$

with boundary conditions

$$
-K_2(\eta)\Phi_y=0, \qquad y=-\tfrac{1}{\beta}, \tag{3.18}
$$

$$
\begin{aligned}
&-K_2(\eta)\Phi_y+K_1(\eta)\Psi\rho+\frac{K_1(\eta)}{K_2(\eta)}\nu^2\sin^2(\theta_1-\theta_2)(\Phi_z-K_1(\eta)\eta_z\Phi_y)\eta_z \\
&\quad+\nu\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\Psi-\gamma(\rho\sin\theta_2+\nu\eta_z\sin\theta_1)=0, \qquad y=0.
\end{aligned} \tag{3.19}
$$

We also introduce a bifurcation parameter μ_1 by writing $\nu=\nu_0+\mu_1$, where ν_0 is a reference value for ν to be chosen later.

To place equations (3.11)–(3.19) on a rigorous footing, we introduce the spaces

$$H^m_{\mathrm{per}}(S) = \{w \in H^m_{\mathrm{loc}}(\mathbb{R}) : w(z+2\pi) = w(z) \text{ for all } z \in \mathbb{R}\},$$
$$H^m_{\mathrm{per}}(\Sigma) = \{w \in H^m_{\mathrm{loc}}((-\tfrac{1}{\beta},0)\times\mathbb{R}) : w(y,z+2\pi) = w(y,z) \text{ for all } (y,z) \in (-\tfrac{1}{\beta},0)\times\mathbb{R}\}$$

for $m \in \mathbb{N}_0$; recall that $H^1_{\mathrm{per}}(S)$ and $H^2_{\mathrm{per}}(\Sigma)$ are Banach algebras, while the formulae $w \mapsto w|_{y=0}$ and $w \mapsto w|_{y=-\frac{1}{\beta}}$ (for $\beta > 0$) define bounded linear mappings $H^m(\Sigma) \to H^{m-1}(S)$ for $m \in \mathbb{N}$. The following proposition relates to mappings appearing in the above equations (see Bagri & Groves [2, Proposition 2.1] for parts (i), (ii) and Buffoni & Toland [3] for parts (iii), (iv)).

Proposition 3.1.

(i) The formula $(w_1,w_2) \mapsto w_1w_2$ defines bounded bilinear mappings $L^2_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$, $H^1_{\mathrm{per}}(\Sigma)\times L^2_{\mathrm{per}}(S) \to L^2_{\mathrm{per}}(\Sigma)$ and $H^1_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(S) \to H^1_{\mathrm{per}}(\Sigma)$.

(ii) The formula

$$(w_1,w_2) \mapsto \int_{-\frac{1}{\beta}}^0 w_1(\cdot,y)w_2(\cdot,y)\,\mathrm{d}y$$

defines bounded bilinear mappings $L^2_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(\Sigma) \to L^2_{\mathrm{per}}(S)$, $H^1_{\mathrm{per}}(\Sigma)\times L^2_{\mathrm{per}}(\Sigma) \to L^2_{\mathrm{per}}(S)$ and $H^1_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(\Sigma) \to H^1_{\mathrm{per}}(S)$.

(iii) The formulae $\eta \mapsto (1+\beta\eta)^{-1}-1$ (for $\beta > 0$) and $\eta \mapsto (1+\mathrm{e}^y\eta)^{-1}-1$ (for $\beta = 0$) yield mappings $H^3_{\mathrm{per}}(S) \to H^3_{\mathrm{per}}(S)$ and $H^3_{\mathrm{per}}(S) \to H^3_{\mathrm{per}}(\Sigma)$ respectively which are defined and analytic in a neighbourhood of the origin.

(iv) For each $n \in \mathbb{N}$ the formula $\eta \mapsto (1+(\nu_0+\mu)^2\sin^2(\theta_1-\theta_2)\eta_z^2)^{-n}-1$ yields a mapping $\mathbb{R}\times H^3_{\mathrm{per}}(S) \to H^2_{\mathrm{per}}(S)$ which is defined and analytic in a neighbourhood of the origin.

(v) For each $n \in \mathbb{N}$ the formula $(\eta,\rho) \mapsto Q(\eta,\rho)^{\frac{n}{2}}$ yields a mapping $H^3_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S) \to H^2_{\mathrm{per}}(S)$ which is defined and analytic in a neighbourhood of the origin.

Let us now define

$$X = \{v = (\eta,\rho,\Phi,\zeta,\xi,\Psi) \in H^3_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(\Sigma)\times H^1_{\mathrm{per}}(S)\times H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\},$$
$$Z = \{v = (\eta,\rho,\Phi,\zeta,\xi,\Psi) \in H^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(\Sigma)\times L^2_{\mathrm{per}}(S)\times H^1_{\mathrm{per}}(S)\times L^2_{\mathrm{per}}(\Sigma)\}.$$

The following lemma, which is a consequence of the previous proposition, shows that the right-hand sides of equations (3.11)–(3.17) define an analytic mapping $v^{\mu_1}_{\mathrm{H}} : \Lambda_1 \times U \to Z$, where U is a neighbourhood of the origin of X. In the notation of the lemma, these equations define a quasilinear evolutionary system

$$v_x = v^{\mu_1}_{\mathrm{H}}(v) \tag{3.20}$$

with nonlinear boundary conditions given by

$$\Phi_y = F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi), \qquad y = -\tfrac{1}{\beta}, \tag{3.21}$$

$$\Phi_y + \gamma(\rho\sin\theta_2 + \nu_0\eta_z\sin\theta_1) = F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi), \qquad y = 0, \tag{3.22}$$

where

$$
\begin{aligned}
&F^{\mu_1}(\eta,\rho,\Phi,\zeta,\xi,\Psi)\\
&\quad = K_2(\eta)K_3\eta\Phi_y + K_1(\eta)\Psi\rho + (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\Psi\\
&\qquad\quad + \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z - \frac{K_1(\eta)}{K_2(\eta)}\gamma\mu_1\eta_z\sin\theta_1,
\end{aligned}
$$

and we note for later use that

$$
\begin{aligned}
&\mathrm{d}F^{\mu_1}[\eta,\rho,\Phi,\zeta,\xi,\Psi](\tilde\eta,\tilde\rho,\tilde\Phi,\tilde\zeta,\tilde\xi,\tilde\Psi)\\
&\quad = -K_2(\eta)^2K_3^2\eta\Phi_y\tilde\eta + K_2(\eta)K_3(\Phi_y\tilde\eta + \eta\tilde\Phi_y) - K_1(\eta)K_2(\eta)K_3\Psi\rho\tilde\eta + K_1(\eta)(\Psi\tilde\rho + \rho\tilde\Psi)\\
&\qquad + (\nu_0+\mu_1)\cos(\theta_1-\theta_2)\big(-K_1(\eta)K_2(\eta)K_3\eta_z\Psi\tilde\eta + K_1(\eta)(\Psi\tilde\eta_z + \eta_z\tilde\Psi)\big)\\
&\qquad + \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\\
&\qquad\quad \times\Big((\Phi_z - K_1(\eta)\eta_z\Phi_y)\tilde\eta_z + \eta_z\big(\tilde\Phi_z + K_1(\eta)K_2(\eta)K_3\eta_z\Phi_y\tilde\eta - K_1(\eta)(\Phi_y\tilde\eta_z + \eta_z\tilde\Phi_y)\big)\Big)\\
&\qquad - \frac{K_1(\eta)}{K_2(\eta)}\gamma\mu_1\sin\theta_1\tilde\eta_z.
\end{aligned}
$$

This system is reversible; the reverser is given by

$$
\begin{aligned}
&R\big(\eta(z),\rho(z),\Phi(y,z),\zeta(z),\xi(z),\Psi(y,z)\big)\\
&\qquad = \big(\eta(-z),-\rho(-z),-\Phi(y,-z),-\zeta(-z),\xi(-z),\Psi(y,-z)\big).
\end{aligned}
$$

Lemma 3.2. *There exist neighbourhoods U and Λ_1 of the origin in respectively X and $\mathbb{R}$ with the following properties.*

- *(i) The formula $(\mu_1, v) \mapsto v_{\mathrm{H}}^{\mu_1}(v)$, where $v_{\mathrm{H}}^{\mu_1}(v)$ is defined by the right-hand sides of* (3.11)–(3.17) *(with $\nu = \nu_0 + \mu_1$), defines an analytic mapping $\Lambda_1 \times U \to Z$.*
- *(ii) The formula $(\mu_1, v) \mapsto F^{\mu_1}(v)$ defines an analytic mapping $\Lambda_1 \times U \to H^1(\Sigma)$.*
- *(iii) The derivative $\mathrm{d}F^{\mu_1}[v] \in \mathcal{L}(X, H^1(\Sigma))$ has a unique extension $\widetilde{\mathrm{d}F}^{\mu_1}[v] \in \mathcal{L}(Z, L^2(\Sigma))$ which depends analytically upon $(\mu_1, v) \in \Lambda_1 \times U$.*

It remains to confirm that (3.20) has a Hamiltonian structure; for this purpose we use the following lemma, which is proved by direct calculations and Proposition 3.1.

Lemma 3.3.

- *(i) The formula $(\mu_1, v) \mapsto H^{\mu_1}(v)$, where $H^{\mu_1}(v)$ is defined by the right-hand side of* (3.10) *(with $\nu = \nu_0 + \mu_1$), defines an analytic mapping $\Lambda_1 \times U \to \mathbb{R}$.*
- *(ii) The derivative $\mathrm{d}H^{\mu_1}[v] \in X^*$ has a unique extension $\widetilde{\mathrm{d}H}^{\mu_1}[v] \in Z^*$ which depends analytically upon $(\mu_1, v) \in \Lambda_1 \times U$.*

(iii) The formula

$$\widetilde{\mathrm{d}H}^{\mu_1}[v](w) = \langle Jv_{\mathrm{H}}^{\mu_1}(v), w\rangle,$$

where $\langle \cdot\,, \cdot\rangle$ *denotes the* $(L^2_{\mathrm{per}}(S))^6$ *inner product and*

$$J(\eta, \rho, \Phi, \zeta, \xi, \Psi) = (-\zeta, -\xi, -\Psi, \eta, \rho, \Phi),$$

holds for all $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ *and* $w \in Z$*, where*

$$\mathcal{D}_{\mathrm{H}} = \{(\mu_1, v) \in \Lambda_1 \times U : \text{(3.21), (3.22) are satisfied}\},$$

so that the gradient $\nabla H^{\mu_1}(v)$ *exists (and equals* $Jv_{\mathrm{H}}^{\mu_1}(v)$*) for all* $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ *and extends to an analytic function of* $(\mu_1, v) \in \Lambda_1 \times U$*.*

Altogether we conclude that $v_{\mathrm{H}}^{\mu_1}(v) = J^{-1}\nabla H^{\mu_1}(v)$ for $(\mu_1, v) \in \mathcal{D}_{\mathrm{H}}$ defines the Hamiltonian vector field for the Hamiltonian system (Z, Ω, H^{μ_1}), where $\Omega : Z^2 \to \mathbb{R}$ is the constant symplectic 2-form

$$\begin{aligned}
&\Omega\big((\eta_1, \rho_1, \Phi_1, \zeta_1, \xi_1, \Psi_1), (\eta_2, \rho_2, \Phi_2, \zeta_2, \xi_2, \Psi_2)\big)\\
&\quad = \langle J(\eta_1, \rho_1, \Phi_1, \zeta_1, \xi_1, \Psi_1), (\eta_2, \rho_2, \Phi_2, \zeta_2, \xi_2, \Psi_2)\rangle\\
&\quad = \int_S (\zeta_2\eta_1 - \eta_2\zeta_1 + \xi_2\rho_1 - \rho_2\xi_1)\,\mathrm{d}z + \int_\Sigma (\Psi_2\Phi_1 - \Phi_2\Psi_1)\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Note that Ω is the exterior derivative of the parameter-independent 1-form $\omega|_v$ given by

$$\begin{aligned}
\omega|_{(\eta,\rho,\Phi,\zeta,\xi,\Psi)}(\tilde{\eta}, \tilde{\rho}, \tilde{\Phi}, \tilde{\zeta}, \tilde{\xi}, \tilde{\Psi}) &= \int_S (\eta\tilde{\zeta} + \rho\tilde{\xi})\,\mathrm{d}z + \int_\Sigma \Phi\tilde{\Psi}\,\mathrm{d}y\,\mathrm{d}z\\
&= \langle \alpha(\eta, \rho, \Phi, \zeta, \xi, \Psi), (\tilde{\eta}, \tilde{\rho}, \tilde{\Phi}, \tilde{\zeta}, \tilde{\xi}, \tilde{\Psi})\rangle,
\end{aligned}$$

where

$$\alpha(\eta, \rho, \Phi, \zeta, \xi, \Psi) = (0, 0, 0, \eta, \rho, \Phi).$$

The system (3.20)–(3.22) is unsuitable for analysis due to its nonlinear boundary conditions. We proceed by replacing Φ with the new variable

$$\Gamma = \Phi - \partial_y \Delta^{-1} F^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi),$$

where Δ is the Dirichlet Laplacian in Σ. In the notation of the following lemma we find that for $(\eta, \rho, \zeta, \xi, \Psi) \in W$ and $\mu_1 \in \Lambda_1$ the variable $\Phi \in V_1$ satisfies (3.21), (3.22) if and only if the variable $\Gamma \in V_2$ satisfies

$$\Gamma_y = 0, \qquad y = -\tfrac{1}{\beta}, \tag{3.23}$$

$$\Gamma_y + \gamma(\rho\sin\theta_2 + \nu_0\eta_z\sin\theta_1) = 0, \qquad y = 0, \tag{3.24}$$

because

$$\begin{aligned}
\Gamma_y &= \Phi_y - \partial_{yy}\Delta^{-1}F^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi)\\
&= \Phi_y - \partial_{yy}\Delta^{-1}F^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi) - \underbrace{\partial_{zz}\Delta^{-1}F^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi)}_{=0}\\
&= \Phi_y - F^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi)
\end{aligned}$$

for $y = 0$ and $y = -\frac{1}{\beta}$.

Lemma 3.4.

(i) *There exist neighbourhoods* V_1, V_2 *of the origin in* $H^2_{\mathrm{per}}(\mathbb{R})$ *and* W *of the origin in*

$$X_0 = \{(\eta, \rho, \zeta, \xi, \Psi) \in H^3_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(\Sigma)\}$$

such that $\Phi \mapsto \Gamma(\Phi, (\eta, \rho, \zeta, \xi, \Psi), \mu_1)$ *is an analytic diffeomorphism* $V_1 \to V_2$ *which, together with its inverse, depends analytically upon* $(\eta, \rho, \zeta, \xi, \Psi) \in W$ *and* $\mu_1 \in \Lambda_1$.

(ii) *The derivative* $\mathrm{d}_1\Gamma[\Phi, (\eta, \rho, \zeta, \xi, \Psi), \mu_1] \in \mathcal{L}(H^2_{\mathrm{per}}(\Sigma))$ *extends to an isomorphism in* $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma))$ *which, together with its inverse, depends analytically upon* $\Phi \in V_1$, $(\eta, \rho, \zeta, \xi, \Psi) \in W$ *and* $\mu_1 \in \Lambda_1$.

Proof. (i) This result follows by applying the implicit-function theorem to the equation

$$g(\Phi, \Gamma, (\eta, \rho, \zeta, \xi, \Psi), \mu_1) := \Gamma - \Gamma(\Phi, (\eta, \rho, \zeta, \xi, \Psi), \mu_1) = 0.$$

Here we note that g maps (a neighbourhood of the origin in) $H^2_{\mathrm{per}}(\Sigma) \times H^2_{\mathrm{per}}(\Sigma) \times X_0 \times \mathbb{R}$ into $H^2_{\mathrm{per}}(\Sigma)$ (by Lemma 3.2(ii) and the fact that Δ^{-1} belongs to $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma), H^3_{\mathrm{per}}(\Sigma))$), and that $g(0,0,0,0) = 0$ and $\mathrm{d}_1 g[0,0,0,0] = -I$.

(ii) It follows from Lemma 3.2(iii) and the fact that Δ^{-1} belongs to $\mathcal{L}(L^2_{\mathrm{per}}(\Sigma), H^2_{\mathrm{per}}(\Sigma))$ that $\mathrm{d}_1\Gamma[\Phi, (\eta, \rho, \zeta, \xi, \Psi), \mu_1] \in \mathcal{L}(H^2_{\mathrm{per}}(\Sigma))$ extends to an element $\widetilde{\mathrm{d}_1\Gamma}[\Phi, (\eta, \rho, \zeta, \xi, \Psi), \mu_1]$ of $\mathcal{L}(H^1_{\mathrm{per}}(\Sigma))$ which depends analytically upon $\Phi \in V_1$, $(\eta, \rho, \zeta, \xi, \Psi) \in W$ and $\mu_1 \in \Lambda_1$. Obviously $\widetilde{\mathrm{d}_1\Gamma}[0,0,0,0] = I$ is an isomorphism, which is an open property. The analyticity of $\widetilde{\mathrm{d}_1\Gamma}[\cdot]$ implies the analyticity of its inverse. □

It follows from the above lemma that the formula

$$G^{\mu_1}(\eta, \rho, \Phi, \zeta, \xi, \Psi) = (\eta, \rho, \Gamma, \zeta, \xi, \Psi)$$

defines a valid change of variable: it is an analytic diffeomorphism from U to a neighbourhood $\hat{U}$ of the origin in X, the operator $\mathrm{d}G^{\mu_1}[u] \in \mathcal{L}(X)$ extends to an isomorphism $\widetilde{\mathrm{d}G^{\mu_1}}[u] \in \mathcal{L}(Z)$, and G^{μ_1}, $\widetilde{\mathrm{d}G^{\mu_1}}[u]$ and their inverses depend analytically upon $(u, \mu_1) \in U \times \Lambda_1$. The system (3.20)–(3.22) is transformed into

$$\hat{v}_x = \hat{v}^{\mu_1}_{\mathrm{H}}(\hat{v}), \tag{3.25}$$

where

$$\hat{v}^{\mu_1}_{\mathrm{H}}(\hat{v}) = \widetilde{\mathrm{d}G^{\mu_1}}[(G^{\mu_1})^{-1}(\hat{v})](v^{\mu_1}_{\mathrm{H}}((G^{\mu_1})^{-1}(\hat{v})))$$

with linear boundary conditions (3.23), (3.24). Note also that G^{μ_1} and $(G^{\mu_1})^{-1}$ both commute with the reverser R, so that (3.25) inherits the reversibility of equation (3.20).

Writing $(G^{\mu_1})^{-1}$ as K^{μ_1}, one finds that the change of variable transforms (Z, Ω, H^{μ_1}) into the new Hamiltonian system $(Z, \hat{\Omega}^{\mu}, \hat{H}^{\mu})$, where

$$\hat{\Omega}^{\mu_1}|_{\hat{v}}(\hat{v}_1, \hat{v}_2) = \langle \hat{J}^{\mu_1}(\hat{v})\hat{v}_1, \hat{v}_2 \rangle \tag{3.26}$$

with

$$\hat{J}^{\mu_1}(\hat{v}) = \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^* J \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]$$

and

$$\hat{H}^{\mu_1}(\hat{v}) = H^{\mu_1}(K^{\mu_1}(\hat{v})) \tag{3.27}$$

for $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$. In particular

$$\hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v}) = \hat{J}^{\mu_1}(\hat{v})\nabla\hat{H}^{\mu_1}(\hat{v})$$

for $(\mu_1, \hat{v}) \in \hat{\mathcal{D}}_{\mathrm{H}}$, where

$$\hat{\mathcal{D}}_{\mathrm{H}} = \{(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U} : \text{(3.23), (3.24) are satisfied}\}.$$

Note further that $\hat{\Omega}^{\mu_1}|_{\hat{v}}$ is the exterior derivative of the 1-form $\hat{\omega}^{\mu_1}|_{\hat{v}}$ given by

$$\hat{\omega}^{\mu_1}|_{\hat{v}}(\hat{w}) = \langle \hat{\alpha}^{\mu_1}(\hat{v}), \hat{w}\rangle$$

with

$$\hat{\alpha}^{\mu_1}(\hat{v}) = \widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^*(\alpha(K^{\mu_1}(\hat{v}))).$$

The validity of these calculations relies upon the existence of the adjoint operator $\widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^*$, and this assumption is verified in the following result.

Proposition 3.5. *The adjoint operators* $\widetilde{\mathrm{d}G^{\mu_1}}[v]^*$, $\widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}]^* \in \mathcal{L}(Z)$ *exist and depend analytically upon* $(\mu_1, v) \in \Lambda_1 \times U$ *and* $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$.

Proof. The existence of the adjoint $\widetilde{\mathrm{d}G^{\mu_1}}[v]^* \in \mathcal{L}(Z)$ follows by a direct calculation; its components are given by

$$\begin{aligned}
(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\eta = \tilde{\eta} + \int_{-\frac{1}{\beta}}^0 \Bigg\{ \Bigg(& - K_2(\eta)^2 K_3^2 \eta \Phi_y + K_2(\eta) K_3 \Phi_y \\
& - K_1(\eta)K_2(\eta)K_3\Psi\rho - (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)K_2(\eta)K_3\eta_z\Psi \\
& + \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\eta_z K_1(\eta)K_2(\eta)K_3\eta_z\Phi_y\Bigg)\Delta^{-1}(\tilde{\Gamma}_y) \\
& - \Bigg(\Bigg((\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\Psi \\
& \qquad + \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y) \\
& \qquad - \frac{K_1^2(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\eta_z\Phi_y \\
& \qquad - \frac{K_1(\eta)}{K_2(\eta)}\gamma\mu_1\sin\theta_1\Bigg)\Delta^{-1}(\tilde{\Gamma}_y)\Bigg)_z\Bigg\}\,\mathrm{d}y,
\end{aligned}$$

$$(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\rho = \tilde{\rho} + \int_{-\frac{1}{\beta}}^0 K_1(\eta)\Psi\Delta^{-1}(\tilde{\Gamma}_y)\,\mathrm{d}y,$$

$$(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\Gamma = \tilde{\Gamma} - \Bigg(\Bigg(K_2(\eta)K_3\eta - \frac{K_1^2(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)\eta_z^2\Bigg)\Delta^{-1}(\tilde{\Gamma}_y)\Bigg)_y,$$

$$- \left(\frac{K_1(\eta)}{K_2(\eta)} (\nu_0 + \mu_1)^2 \sin^2(\theta_1 - \theta_2) \eta_z \Delta^{-1}(\tilde{\Gamma}_y) \right)_z,$$

$$(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\zeta = \tilde{\zeta},$$

$$(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\xi = \tilde{\xi},$$

$$(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*(\tilde{v}))_\Psi = \tilde{\Psi} + \left(K_1(\eta)\rho + (\nu_0 + \mu_1)\cos(\theta_1 - \theta_2) K_1(\eta)\eta_z \right) \Delta^{-1}(\tilde{\Gamma}_y),$$

from which the analyticity of $\widetilde{\mathrm{d}G^{\mu_1}}[v]$ also follows by Proposition 3.1. (Note that the formula $(\Delta^{-1})^* = \Delta^{-1}$ has been used in this calculation.)

Observe that $\widetilde{\mathrm{d}G^{\mu_1}}[v]^*$ is an isomorphism because $\widetilde{\mathrm{d}G^0}[0]^* = I$ is obviously an isomorphism, which is an open property. Furthermore the analytic dependence of $\widetilde{\mathrm{d}G^{\mu_1}}[v]^*$ upon $(\mu_1, v) \in \Lambda_1 \times U$ implies the same of its inverse, and the calculation

$$\begin{aligned} \langle \widetilde{\mathrm{d}G^{\mu_1}}[v]^{-1}(v_1), v_2 \rangle &= \langle \widetilde{\mathrm{d}G^{\mu_1}}[v]^{-1} v_1, \widetilde{\mathrm{d}G^{\mu_1}}[v]^* (\widetilde{\mathrm{d}G^{\mu_1}}[v]^*)^{-1}(v_2) \rangle \\ &= \langle \widetilde{\mathrm{d}G^{\mu_1}}[v] \widetilde{\mathrm{d}G^{\mu_1}}[v]^{-1}(v_1), (\widetilde{\mathrm{d}G^{\mu_1}}[v]^*)^{-1}(v_2) \rangle \\ &= \langle v_1, (\widetilde{\mathrm{d}G^{\mu_1}}[v]^*)^{-1}(v_2) \rangle \end{aligned}$$

shows that $(\widetilde{\mathrm{d}G^{\mu_1}}[v]^{-1})^*$ exists and equals $(\widetilde{\mathrm{d}G^{\mu_1}}[v]^*)^{-1}$. The proof is completed by noting that $\widetilde{\mathrm{d}K^{\mu_1}}[\hat{v}] = \widetilde{\mathrm{d}G^{\mu_1}}[v]^{-1}$ with $v = (G^{\mu_1})^{-1}(\hat{v})$. □

4 Application of Lyapunov centre theory to hydroelastic waves

In this section we apply Theorem 1.2 to the spatial dynamics formulation

$$\hat{v}_x = \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})$$

for hydroelastic waves derived in Section 3. For this purpose we define

$$\begin{aligned} X = \{\hat{v} = (\eta, \rho, \Gamma, \zeta, \xi, \Psi) \in\ & H^3_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(S) \times H^2_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(\Sigma) : \\ & \Gamma_y\big|_{y=-\frac{1}{\beta}} = 0,\ \Gamma_y\big|_{y=0} + \gamma(\rho \sin\theta_2 + \nu_0 \eta_z \sin\theta_1) = 0\}, \\ Z = \{v = (\eta, \rho, \Gamma, \zeta, \xi, \Psi) \in\ & H^2_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(S) \times H^1_{\mathrm{per}}(S) \times L^2_{\mathrm{per}}(\Sigma)\} \end{aligned}$$

(note the modification to the space X) and consider $\hat{v}_{\mathrm{H}}^{\mu_1}$ as the Hamiltonian vector field for the Hamiltonian system $(Z, \hat{\Omega}^{\mu_1}, \hat{H}^{\mu_1})$, where $\hat{\Omega}^{\mu_1}$ and $\hat{H}^{\mu_1}$ are defined in equations (3.26) and (3.27) and $\mathcal{D}_H = \Lambda_1 \times \hat{U}$ is a neighbourhood of the origin in $\mathbb{R} \times X$. Defining $L^{\mu_1} = \mathrm{d}\hat{v}_{\mathrm{H}}^{\mu_1}[0]$ and $N^{\mu_1}(\hat{v}) = \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v}) - L^{\mu_1}\hat{v}$, one can write Hamilton's equations as

$$\hat{v}_x = L^{\mu_1}\hat{v} + N^{\mu_1}(\hat{v}), \tag{4.1}$$

where in particular $L := L^0$ is given by the explicit formula

$$L\begin{pmatrix}\eta\\ \rho\\ \Gamma\\ \zeta\\ \xi\\ \Psi\end{pmatrix} = \begin{pmatrix}\rho\\ \xi - \nu_0^2\eta_{zz} - 2\nu_0\cos(\theta_1-\theta_2)\rho_z\\ \Psi - \nu_0\cos(\theta_1-\theta_2)\Gamma_z\\ \nu_0^2\xi_{zz} + \eta - \gamma\nu_0\sin\theta_1\Gamma_z|_{y=0}\\ -\zeta + \gamma\sin\theta_2\Gamma|_{y=0} - 2\nu_0\cos(\theta_1-\theta_2)\xi_z\\ -\Gamma_{yy} - \nu_0^2\sin^2(\theta_1-\theta_2)\Gamma_{zz} - \nu_0\cos(\theta_1-\theta_2)\Psi_z\end{pmatrix},$$

which is readily calculated from (3.11)–(3.17) since the change of variable used to linearise the boundary condition is near-identity. Equation (4.1) evidently satisfies hypothesis (H1).

Furthermore, the reverser R clearly satisfies $R^* = R$ and

$$H^{\mu_1}(Rv) = H^{\mu_1}(v), \quad R^*\alpha^{\mu_1}(Rv) = -\alpha^{\mu_1}(v), \quad R^*J^{\mu_1}(Rv)R = -J^{\mu_1}(v)$$

for all $(\mu_1, v) \in \Lambda_1 \times U$; since R commutes with G^{μ_1} and K^{μ_1} we conclude that

$$\hat{H}^{\mu_1}(R\hat{v}) = \hat{H}^{\mu_1}(\hat{v}), \quad R^*\hat{\alpha}^{\mu_1}(R\hat{v}) = -\hat{\alpha}^{\mu_1}(\hat{v}), \quad R^*\hat{J}^{\mu_1}(R\hat{v})R = -\hat{J}^{\mu_1}(\hat{v})$$

for all $(\mu_1, \hat{v}) \in \Lambda_1 \times \hat{U}$. Hypothesis (H2) is therefore also satisfied.

4.1 Purely imaginary spectrum

The next step is to examine the purely imaginary spectrum of the linear operator L. This task is readily accomplished by using Fourier-series representations

$$\hat{v}(y,z) = \sum_{k\in\mathbb{Z}} \hat{v}_k(y)\mathrm{e}^{\mathrm{i}kz}, \qquad \hat{v}^\star(y,z) = \sum_{k\in\mathbb{Z}} \hat{v}_k^\star(y)\mathrm{e}^{\mathrm{i}kz}$$

for $\hat{v} \in X$ and $\hat{v}^\star \in Z$ and examining the resulting decoupled spectral problems for each Fourier mode. We begin with the following lemma, which is proved by well-established methods for spatial dynamics problems (see Groves & Haragus [5], Bagri & Groves [2] and the references therein).

Lemma 4.1.

(i) Suppose that $s \in \mathbb{R}$ and $k \in \mathbb{Z}$ are not both zero. The imaginary number $\mathrm{i}s$ *is a mode k eigenvalue of L if and only if*

$$(1+\sigma_k^4)\sigma_k - \frac{\gamma^2(k\nu_0\sin\theta_1 + s\sin\theta_2)^2}{\tanh(\beta^{-1}\sigma_k)} = 0, \tag{4.2}$$

where

$$\sigma_k^2 = s^2 + 2k\nu_0 s\cos(\theta_1-\theta_2) + k^2\nu_0^2 > 0,$$

and in this case the corresponding eigenvector is $\hat{v}_{k,s}\mathrm{e}^{\mathrm{i}kz}$, *where*

$$\hat{v}_{k,s} = \begin{pmatrix} \dfrac{\mathrm{i}\gamma b_k}{1+\sigma_k^4} \\ -\dfrac{s\gamma b_k}{1+\sigma_k^4} \\ \frac{1}{2}\mathrm{e}^{-\sigma_k y}(1-t_k) + \frac{1}{2}\mathrm{e}^{\sigma_k y}(1+t_k) \\ \gamma\sin\theta_2 - \dfrac{\gamma\sigma_k^2 a_{2k} b_k}{1+\sigma_k^4} \\ -\dfrac{\mathrm{i}\gamma\sigma_k^2 b_k}{1+\sigma_k^4} \\ \frac{1}{2}\mathrm{i}a_k\Big(\mathrm{e}^{-\sigma_k y}(1-t_k) + \mathrm{e}^{\sigma_k y}(1+t_k)\Big) \end{pmatrix} \tag{4.3}$$

and

$$t_k = \tanh(\beta^{-1}\sigma_k), \quad a_k = s + k\nu_0\cos(\theta_1-\theta_2), \quad b_k = k\nu_0\sin\theta_1 + s\sin\theta_2 > 0.$$

This eigenvalue has a Jordan chain of length at least 2 *with generalised eigenvector* $\hat{w}_{k,s}\mathrm{e}^{\mathrm{i}kz}$, *where*

$$\hat{w}_{k,s} = \begin{pmatrix} \dfrac{\gamma\sin\theta_2}{1+\sigma_k^4} - 2\gamma a_k b_k\left(\dfrac{2\sigma_k^2}{(1+\sigma_k^4)^2} + \dfrac{c_k}{2\sigma_k^2(1+\sigma_k^4)}\right) \\ \dfrac{\mathrm{i}\gamma b_k}{1+\sigma_k^4} + \dfrac{\mathrm{i}s\gamma\sin\theta_2}{1+\sigma_k^4} - 2\mathrm{i}\gamma s a_k b_k\left(\dfrac{2\sigma_k^2}{(1+\sigma_k^4)^2} + \dfrac{c_k}{2\sigma_k^2(1+\sigma_k^4)}\right) \\ \dfrac{\mathrm{i}a_k}{2\sigma_k}\big((1-t_k)y\mathrm{e}^{-\sigma_k y} - (1+t_k)y\mathrm{e}^{\sigma_k y}\big) + \dfrac{\mathrm{i}c_k a_k}{2\sigma_k^2}(\mathrm{e}^{-\sigma_k y} + \mathrm{e}^{\sigma_k y}) \\ \mathrm{i}\left(\dfrac{c_k a_k}{\sigma_k^2} + \dfrac{\sigma_k^2 a_{2k}}{1+\sigma_k^4}\right)\gamma\sin\theta_2 + \dfrac{\mathrm{i}\gamma\sigma_k^2 b_k}{1+\sigma_k^4} - 2\mathrm{i}\gamma a_k a_{2k} b_k\left(\dfrac{2\sigma_k^4}{(1+\sigma_k^4)^2} + \dfrac{c_k-2}{2(1+\sigma_k^4)}\right) \\ -\dfrac{\sigma_k^2}{1+\sigma_k^4}\gamma\sin\theta_2 + 2\gamma a_k b_k\left(\dfrac{2\sigma_k^4}{(1+\sigma_k^4)^2} + \dfrac{c_k-2}{2(1+\sigma_k^4)}\right) \\ \left(\frac{1}{2}(1-t_k) - \dfrac{c_k a_k^2}{2\sigma_k^2} - \dfrac{(1-t_k)a_k^2}{2\sigma_k}y\right)\mathrm{e}^{-\sigma_k y} + \left(\frac{1}{2}(1+t_k) - \dfrac{c_k a_k^2}{2\sigma_k^2} + \dfrac{(1+t_k)a_k^2 y}{2\sigma_k}\right)\mathrm{e}^{\sigma_k y} \end{pmatrix}$$

and

$$c_k = 2\beta^{-1}\sigma_k\operatorname{cosech}(2\beta^{-1}\sigma_k),$$

if either

(a) $\beta > 0$, $s = a\beta$, $\nu_0 = \tilde{\nu}_0\beta$, $(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2 \neq 0$ *and* (β,γ) *lies on a point of the curve*

$$C_k = \{(\beta_k(a), \gamma_k(a)) : a \in (0,\infty)\},$$

where

$$\beta_k^4(a) = \frac{1}{\tilde{\sigma}_k^4}\cdot\frac{2\sin\theta_2\tilde{\sigma}_k^2 - (1+\tilde{c}_k)\tilde{a}_k\tilde{b}_k}{(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2},$$

$$\gamma_k^2(a) = \frac{(1+\beta_k^4(a)\tilde{\sigma}_k^4)\tilde{\sigma}_k \tanh(\tilde{\sigma}_k)}{\beta_k(a)\tilde{b}_k^2}$$

and

$$\begin{aligned}
\tilde{\sigma}_k^2 &= a^2 + 2k\tilde{\nu}_0 a\cos(\theta_1-\theta_2) + k^2\tilde{\nu}_0^2,\\
\tilde{a}_k &= a + k\tilde{\nu}_0\cos(\theta_1-\theta_2),\\
\tilde{b}_k &= a\sin\theta_2 + k\tilde{\nu}_0\sin\theta_1,\\
\tilde{c}_k &= 2\tilde{\sigma}_k \operatorname{cosech}(2\tilde{\sigma}_k);
\end{aligned}$$

(b) $\beta > 0$, $s = a\beta$, $\nu_0 = \tilde{\nu}_0\beta$ *and* $(5+\tilde{c}_k)\tilde{a}_k\tilde{b}_k - 2\sin\theta_2\tilde{\sigma}_k^2 = 0$, *which implies that* $\theta_2 = 0$ *and* $a = -k\tilde{\nu}_0\cos(\theta_1)$*;*

(c) $\beta = 0$ *and*

$$2\sin\theta_2 - \frac{4\sigma_k^2 a_k b_k}{1+\sigma_k^4} = \frac{a_k b_k}{\sigma_k^2}.$$

(ii) Suppose $\beta > 0$*. Zero is a mode* 0 *eigenvalue of* L *with a Jordan chain of length 2 if* $\gamma^{-2} \neq \beta\sin^2\theta_2$ *and length* 4 *if* $\gamma^{-2} = \beta\sin^2\theta_2$*; the generalised eigenvectors are*

$$\hat{f}_1 = \begin{pmatrix} 0\\ 0\\ 1\\ \gamma\sin\theta_2\\ 0\\ 0\end{pmatrix}, \quad \hat{f}_2 = \begin{pmatrix} \gamma\sin\theta_2\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}, \quad \hat{f}_3 = \begin{pmatrix} 0\\ \gamma\sin\theta_2\\ -\frac{1}{2}y^2 - \beta^{-1}y\\ 0\\ 0\\ 0\end{pmatrix}, \quad \hat{f}_4 = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ \gamma\sin\theta_2\\ -\frac{1}{2}y^2 - \beta^{-1}y\end{pmatrix},$$

where $L\hat{f}_1 = 0$*,* $L\hat{f}_2 = \hat{f}_1$ *and* $L\hat{f}_3 = \hat{f}_2$*,* $L\hat{f}_4 = \hat{f}_3$ *if* $\gamma^{-2} = \beta\sin^2\theta_2$*.*

(iii) Suppose $\beta = 0$ *and that* $s = 0$ *does not solve* (4.2) *for any* $k \in \mathbb{Z}\setminus\{0\}$ *(so that zero is not a mode* k *eigenvalue for any* $k \in \mathbb{Z}\setminus\{0\}$*). Zero is not a mode* 0 *eigenvalue of* L*, which instead has essential spectrum at the origin. More precisely, the equation* $L\hat{v} = \hat{v}^\star$ *has a unique solution for each* $\hat{v}^\star \in Z$ *which satisfies the regularity requirement*

$$\int_{-\infty}^{y}\int_{-\infty}^{t}\Psi_0^\star(s)\,\mathrm{d}s\,\mathrm{d}t,\ \int_{-\infty}^{y}\Psi_0^\star(t)\,\mathrm{d}t \in L^2(-\infty,0)$$

and compatibility condition

$$\int_{-\infty}^{0}\Psi_0^\star(t)\,\mathrm{d}t - \gamma\sin\theta_2\eta_0^\star = 0.$$

This solution satisfies the estimate

$$\|\hat{v} - [\![\hat{v}]\!]_0\|_X \lesssim \|\hat{v}^\star\|_Z$$

and its 0*th Fourier component is given by the formula*

$$\begin{pmatrix} \hat{\eta}_0 \\ \hat{\rho}_0 \\ \hat{\Gamma}_0 \\ \hat{\zeta}_0 \\ \hat{\xi}_0 \\ \hat{\Psi}_0 \end{pmatrix} = \begin{pmatrix} \zeta_0^\star \\ \eta_0^\star \\ -\int_{-\infty}^{y}\int_{-\infty}^{t} \Psi_0^\star(s)\,\mathrm{d}s\,\mathrm{d}t \\ -\xi_0^\star - \gamma\sin\theta_2 \int_{-\infty}^{0}\int_{-\infty}^{t} \Psi_0^\star(s)\,\mathrm{d}s\,\mathrm{d}t \\ \rho_0^\star \\ \Gamma_0^\star \end{pmatrix}.$$

(iv) Suppose that $s \in \mathbb{R} \setminus \{0\}$ *does not satisfy* (4.2) *for any* $k \in \mathbb{Z}$*. The imaginary number* $\mathrm{i}s$ *belongs to the resolvent set of* L*.*

(v) The resolvent estimates

$$\|(\mathrm{i}sI - L)^{-1}\|_{Z\to X} \lesssim 1, \qquad \|(\mathrm{i}sI - L)^{-1}\|_{Z\to Z} \lesssim \frac{1}{|s|}$$

hold uniformly over all sufficiently large values of $|s|$*.*

We proceed by interpreting equation (4.2) geometrically. Let

$$\ell_1 = s\sin\theta_2 + \nu_0 k \sin\theta_1, \tag{4.4}$$

$$\ell_2 = -s\cos\theta_2 - \nu_0 k \cos\theta_1, \tag{4.5}$$

and note that $\ell_1^2 + \ell_2^2 = \sigma_k^2$, so that (4.2) can be written as

$$\mathcal{D}(\ell_1,\ell_2) := (1 + (\ell_1^2+\ell_2^2)^2)\sqrt{\ell_1^2+\ell_2^2}\tanh\left(\beta^{-1}\sqrt{\ell_1^2+\ell_2^2}\right) - \gamma^2\ell_1^2 = 0. \tag{4.6}$$

A mode k purely imaginary eigenvalue $\mathrm{i}s$ (with $(k,s) \neq (0,0)$) therefore corresponds to an intersection in the (ℓ_1,ℓ_2)-plane of the dispersion curve

$$C_{\mathrm{dr}} = \{(\ell_1,\ell_2) \in \mathbb{R}^2 \setminus \{(0,0)\} : \mathcal{D}(\ell_1,\ell_2) = 0\}$$

with the straight line S_k defined by equations (4.4), (4.5). (The solution $(\ell_1,\ell_2) = (0,0)$ of $\mathcal{D}(\ell_1,\ell_2) = 0$ is excluded since it corresponds to $(k,s) = (0,0)$.)

The dispersion curve C_{dr} is described parametrically by

$$C_{\mathrm{dr}} := \left\{(\ell_1,\ell_2) \in \mathbb{R}^2 : \ell_1^2 = \frac{(1+a^4)a}{\gamma^2}\tanh(\beta^{-1}a),\ \ell_2^2 = a^2 - \frac{(1+a^4)a}{\gamma^2}\tanh(\beta^{-1}a),\ a > 0\right\};$$

its shape is shown in Figure 2 (a, insets) in the indicated regions of the (β,γ)-parameter plane. The delimiting curves are

$$D_1 = \{(\beta_0(a),\gamma_0(a))\big|_{k=0,\,\theta_2=\frac{\pi}{2}} : a \in (0,\infty)\},$$

at each point of which the equation $\mathcal{D}(a\beta, 0) = 0$ has double roots $\pm a$, and

$$D_2 = \{(\beta, \beta^{-1/2}) : \beta \geq 0\},$$

at each point of which the equation $\mathcal{D}(\ell_1, 0) = 0$ has a double zero root. We find that $C_{\rm dr} = \emptyset$ in the region below the curve D_1. In the region between the curves D_1 and D_2 the equation $\mathcal{D}(\ell_1, 0) = 0$ has two pairs of simple nonzero roots $\pm\ell_1^{(1)}$, $\pm\ell_1^{(2)}$; the branches of $C_{\rm dr}$ intersect the ℓ_1 axis vertically at the points $(\pm\ell_1^{(1)}, 0)$ and $(\pm\ell_1^{(2)}, 0)$. Notice the qualitative difference in the shape of $C_{\rm dr}$ in the two subregions. In the upper subregion the portion $C_{\rm dr}^+$ of $C_{\rm dr}$ in the positive quadrant has two points of inflection (it is concave to the left of the first and to the right of the second and convex in between); passing into the lower subregion, one finds that the two points of inflection merge and disappear, so that $C_{\rm dr}^+$ becomes concave. The points $(\pm\ell_1^{(1)}, 0)$ approach the origin as one passes through D_2 from below to above, as does the left point of inflection on $C_{\rm dr}^+$. In the region above D_2 the branches of $C_{\rm dr}$ intersect the ℓ_1 axis vertically at the points $(\pm\ell_1^{(2)}, 0)$, while in a neighbourhood of $(0,0)$ they have the limiting behaviour

$$\ell_2^2 \sim \left(\gamma^2\beta - 1\right)\ell_1^2$$

as $\ell_1 \to 0$ and therefore make angles $\pm\arctan\sqrt{\gamma^2\beta - 1}$ with the ℓ_1 axis at the origin. The subcurve $C_{\rm dr}^+$ has a single point of inflection, to the left and right of which it is respectively convex and concave.

For given ν, θ_1 and θ_2 the lines S_k in the (ℓ_1, ℓ_2)-plane are parallel, equidistant and form an angle θ_2 with the positive ℓ_2-axis. They intersect the line

$$T = \{(\ell_1, \ell_2) \in \mathbb{R}^2 \ : \ \ell_1 = \sin\theta_1\, a,\ \ell_2 = -\cos\theta_1\, a,\ a \in \mathbb{R}\}$$

(which passes through the origin and makes an angle θ_1 with the positive ℓ-axis) at the points $P_k = (\sin\theta_1\, k\nu_0, -\cos\theta_1\, k\nu_0)$, $k \in \mathbb{Z}$ (see Figure 2(b)). The number of points in the set $S_0 \cap C_{\rm dr}$ depends only upon (β, γ), which determines the shape of $C_{\rm dr}$, and θ_2, which determines the slope of each line S_k. Furthermore, for fixed β, γ and θ_2 the number of points in the sets $S_k \cap C_{\rm dr}$, $k = \pm 1, \pm 2, \ldots$ depends only upon ν_0, which determines the distance between the lines S_k. At each fixed point of the (β, γ)-parameter plane the number of purely imaginary eigenvalues of the linear operator L therefore depends upon the two parameters θ_2 and ν_0; the third parameter θ_1, which specifies the slope of the line T, influences only the values of these eigenvalues and their relative positions on the imaginary axis: the imaginary part of a purely imaginary eigenvalue corresponding to an intersection of S_k and $C_{\rm dr}$ is the value of S_0 in the (S_0, T)-coordinate system at the intersection (the signed distance between the intersection and the point P_n). The geometric multiplicity of the eigenvalue $\mathrm{i}s$ is given by the number of distinct lines in the family S_k that intersect $C_{\rm dr}$ at this parameter value, and a tangent intersection between S_k and $C_{\rm dr}$ indicates that each eigenvector in mode k has an associated Jordan chain of length at least 2. Finally, notice that the sets $S_k \cap C_{\rm dr}$ and $S_{-k} \cap C_{\rm dr}$ have the same cardinality: the purely imaginary number $\mathrm{i}s$ is a mode k eigenvalue if and only if the purely imaginary number $-\mathrm{i}s$ is a mode $-k$ eigenvalue.

Figure 4 illustrates how the purely imaginary mode 0 eigenvalues depend upon (β, γ), nonzero pairs $\pm\mathrm{i}s$ of which satisfy

$$(1 + s^4)|s| - \frac{\gamma^2 \sin^2\theta_2 s^2}{\tanh(\beta^{-1}|s|)} = 0.$$

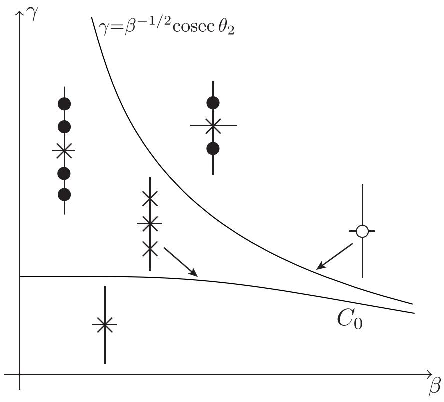

Figure 4: Mode 0 eigenvalues in the (β, γ)-parameter plane. Solid dots, crosses and hollow dots represent eigenvalues with Jordan chains of length 1, 2 and 4; the zero eigenvalue becomes essential spectrum at $\beta = 0$.

The delimiting curves are C_0 and $\{(\beta, \beta^{-1/2}\,\text{cosec}\,\theta_2) : \beta > 0\}$, points of which are associated with respectively non-zero eigenvalues $\pm \mathrm{i} a\beta_0(a)$ with a Jordan chain of length 2 and a zero eigenvalue with a Jordan chain of length 4.

4.2 Parameter selection

We now choose β, γ and θ_2 such that S_0 does not intersect C_{dr} (so that (β, γ) lies in the region above the curve C_0 in Figure 4), and ν_0 and θ_1 such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system, while S_k does not intersect C_{dr} for $k = \pm 2, \pm 3, \ldots$ (see Figure 3). In this configuration L has two mode 1 eigenvalues $\pm \mathrm{i}s$, two mode -1 eigenvalues $\pm \mathrm{i}s$, so that $\pm \mathrm{i}s$ are geometrically and algebraically double eigenvalues of L with eigenvectors $\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}$ and $\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z}$, $\hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z}$, and, for $\beta > 0$, a mode 0 zero eigenvalue with a Jordan chain $\hat{f}_1$, $\hat{f}_2$ of length 2. Note that L^{μ_1} has two mode 1 eigenvalues $\mathrm{i}s_1^{\mu_1}$, $-\mathrm{i}s_{-1}^{\mu_1}$ and two mode -1 eigenvalues $\mathrm{i}s_{-1}^{\mu_1}$, $-\mathrm{i}s_1^{\mu_1}$ which satisfy

$$g(s_{\pm 1}^{\mu_1}, \nu_0 + \mu_1, \pm 1) = 0, \tag{4.7}$$

where

$$g(t, \nu, \pm 1) = (1 + (\sigma_{\pm 1})^4)\sigma_{\pm 1} - \frac{\gamma^2(\pm \nu^2 \sin\theta_1 + t\sin\theta_2)^2}{\tanh(\beta^{-1}\sigma_{\pm 1})},$$
$$(\sigma_{\pm 1})^2 = t^2 \pm 2\nu t\cos(\theta_1 - \theta_2) + \nu^2$$

and $s_{\pm 1}^0 = s$ (the corresponding eigenvectors are given by $\mathrm{d}G[0](v_{1,\pm s_{\pm}^{\mu_1}})$ and $\mathrm{d}G[0](v_{-1,\pm s_{\pm}^{\mu_1}})$, where $v_{1,\pm s_{\pm}^{\mu_1}}$, $v_{-1,\pm s_{\pm}^{\mu_1}}$ are defined by the right-hand side of equation (4.3) with ν_0 replaced by $\nu_0 + \mu_1$). Differentiating (4.7) with respect to μ_1, we obtain the formulae

$$\frac{\mathrm{d}}{\mathrm{d}\mu_1} s_{\pm 1}^{\mu_1} = -\frac{g_\nu(s_{\pm 1}^{\mu}, \nu_0 + \mu, \pm 1)}{g_t(s_{\pm 1}^{\mu}, \nu_0 + \mu, \pm 1)}$$

which imply that

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^{\mu_1} - s_{-1}^{\mu_1})|_{\mu_1 = 0} = -\frac{g_\nu(s, \nu_0, 1)}{g_t(s, \nu_0, 1)} + \frac{g_\nu(s, \nu_0, -1)}{g_t(s, \nu_0, -1)}$$

$$= \frac{1}{g_t(s,\nu_0,1)g_t(s,\nu_0,-1)} \begin{vmatrix} g_t(s,\nu_0,1) & g_t(s,\nu_0,-1) \\ g_\nu(s,\nu_0,1) & g_\nu(s,\nu_0,-1) \end{vmatrix}.$$

It follows that $\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^\mu - s_{-1}^\mu)|_{\mu=0} = 0$ if and only if $\nabla g(s,\nu_0,1),\ \nabla g(s,\nu_0,-1)$ are parallel, in other words if and only if the solution curves of

$$g(t,\nu,1) = 0,$$
$$g(t,\nu,-1) = 0,$$

intersect tangentially at the point (s,ν_0) in the (t,ν)-plane. This observation indicates that generically $\frac{\mathrm{d}}{\mathrm{d}\mu}(s_1^\mu - s_{-1}^\mu)|_{\mu=0} \neq 0$.

Noting that

$$\hat{v}_{1,-s}\mathrm{e}^{\mathrm{i}z} = \overline{\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}} = R(\hat{v}_{-1,s}\mathrm{e}^{-\mathrm{i}z}), \qquad \hat{v}_{-1,-s}\mathrm{e}^{-\mathrm{i}z} = \overline{\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z}} = R(\hat{v}_{1,s}\mathrm{e}^{\mathrm{i}z})$$

and $R\hat{f}_1 = -\hat{f}_1$, $R\hat{f}_2 = \hat{f}_2$, we find that there exists parameters such that hypotheses (H3)–(H6) are satisfied (with (H4)(ii) for $\beta = 0$ and (H4)(iii) for $\beta > 0$). Hypothesis (H7) follows from the observations that $(G^{\mu_1})^{-1}(\hat{f}_1) = \hat{f}_1$ and that $\hat{J}^{\mu_1}(\hat{v})$, $\hat{H}^{\mu_1}(\hat{v})$ depend upon Γ and ζ only through Γ_y and $\zeta - \gamma \sin\theta_1 \Gamma|_{y=0}$, while hypothesis (H8) is verified in Section 4.3 below.

Applying Theorem 1.2 thus yields a family of doubly periodic waves whose periodic cells are small perturbations of the basic periodic cell defined by the periods $2\pi/\nu_0$, $2\pi/s$ and angle $\theta_2 - \theta_1$ between the periodic directions. The following lemma shows that it is in fact possible to choose the basic periodic cell (and value of β) arbitrarily and adjust the value of γ and the angle θ_1 to ensure that Theorem 1.2 applies in this configuration (under the additional hypothesis that S_k does not intersect C_{dr} for $k \neq \pm 1$) .

Lemma 4.2. *Choose β, s, ν_0 and $\theta_2 - \theta_1$. There exist θ_1 and γ such that S_1 and S_{-1} each intersect C_{dr} in points with coordinates $(\pm s, \nu_0)$ and $(\pm s, -\nu_0)$ in the (S_0, T)-coordinate system.*

Proof. The lines S_1 and S_{-1} intersect C_{dr} at respectively (s,ν_0) and $(s,-\nu_0)$ if and only if

$$\gamma^2 b_1^2 = \tanh(\beta^{-1}\sigma_1)(1+\sigma_1^4)\sigma_1,$$
$$\gamma^2 b_{-1}^2 = \tanh(\beta^{-1}\sigma_{-1})(1+\sigma_{-1}^4)\sigma_{-1},$$

and in this case they also intersect C_{dr} at respectively $(-s,-\nu_0)$ and $(-s,\nu_0)$ (see the remarks at the end of Section 4.1). Let e_x, e_z and i be unit vectors in the x, z and x_1 directions (see Figure 1) and set

$$\ell_1 = se_z + \nu_0 e_x \qquad \ell_2 = -se_z + \nu_0 e_x, \tag{4.8}$$

so that $\sigma_1^2 = |\ell_1|^2$, $\sigma_{-1} = |\ell_2|^2$, $i \cdot \ell_1 = b_1$, $i \cdot \ell_2 = b_{-1}$; in this notation our task is to find a solution of the equations

$$(v\cdot\ell_1)^2 = \tanh(\beta^{-1}|\ell_1|)(1+|\ell_1|^4)|\ell_1|, \tag{4.9}$$
$$(v\cdot\ell_2)^2 = \tanh(\beta^{-1}|\ell_2|)(1+|\ell_2|^4)|\ell_2|, \tag{4.10}$$

of the form $v = \gamma i$. Let ψ denote the angle between ℓ_1 and ℓ_2; equation (4.8) shows that rotating e_x and e_z through the same angle θ, that is changing θ_1 and θ_2 by θ, causes ℓ_1 and ℓ_2 to rotate through θ and thus does not change ψ.

Observe that the equations

$$v \cdot n_1 = \pm\alpha_1,$$
$$v \cdot n_2 = \pm\alpha_2,$$

where $n_1, n_2 \in \mathbb{R}^2$ are linearly independent and $\alpha_1, \alpha_2 > 0$, represent two pairs of parallel lines which intersect in four points. Each of these points v satisfies

$$|v| = \frac{|\alpha_2 n_1 \pm \alpha_1 n_2|}{|n_1||n_2|},$$

so that $|v|$ depends only upon α_1, α_2, $|n_1|$, $|n_2|$ and the angle between n_1 and n_2. Using this result we find that the solution set to equations (4.9), (4.10) consists of four points. Let v be one of these points, and note that $|v|$ depends only upon $|\ell_1|$, $|\ell_2|$ and ψ (since the right-hand sides of (4.9), (4.10) depend only upon $|\ell_1|$ and $|\ell_2|$.) By rotating ℓ_1 and ℓ_2 through a suitably chosen angle θ, that is changing θ_1 and θ_2 by θ, we can arrange that $v = |v|i$ and hence $v = \gamma i$ by setting $\gamma = |v|$.

□

4.3 Verification of hypothesis (H8)

It remains to verify that hypothesis (H8) is satisfied when $\beta = 0$. The condition is that the equation

$$Lv^\dagger = \hat{J}^0(0)^{-1}[\hat{J}^{\mu_1}(\hat{v})\big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\big) + \hat{J}^0(0)L\hat{v}]_0, \tag{4.11}$$

has a unique solution $v^\dagger \in X$ which depends smoothly upon $(\hat{v}, \mu_1, \mu_2) \in \mathcal{U} \times \Lambda_1 \times \Lambda_2$. Since

$$\hat{J}^0(0)^{-1}[\hat{J}^0(0)L\hat{v}]_0 = L[\hat{v}]_0$$

one can rewrite equation (4.11) as

$$L\big(v^\dagger - [\hat{v}]_0\big) = \hat{J}^0(0)^{-1}[\hat{J}^{\mu_1}(\hat{v})\big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\big)]_0. \tag{4.12}$$

In view of equation (4.12) and Lemma 4.1(iii) our task is to show that

$$(\hat{v}, \mu_1, \mu_2) \mapsto \int_{-\infty}^{y} [\![[\hat{w}_\Gamma]_0]\!]_0, \qquad (\hat{v}, \mu_1, \mu_2) \mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[\hat{w}_\Gamma]_0]\!]_0 \,\mathrm{d}s\,\mathrm{d}t, \tag{4.13}$$

where

$$\hat{w} = \hat{J}^{\mu_1}(\hat{v})\big((\kappa_0 + \mu_2)\hat{v}_\tau - \hat{v}_{\mathrm{H}}^{\mu_1}(\hat{v})\big),$$

are smooth mappings $\mathcal{U} \times \Lambda_1 \times \Lambda_2 \to L^2(-\infty, 0)$ and that

$$\int_{-\infty}^{0} [\![[\hat{w}_\Gamma]_0]\!]_0 \,\mathrm{d}y = -[\![[\hat{w}_\zeta]_0]\!]_0 \gamma \sin\theta_2 \tag{4.14}$$

for each $(\hat{v}, \mu_1, \mu_2) \in \mathcal{U}\times\Lambda_1\times\Lambda_2$. Here we use the symbols $[\cdot]_0$ and $[\![\cdot]\!]_0$ to denote the projections onto the 0th Fourier modes in the τ and z variables respectively. This task is accomplished in Lemma 4.4 below with the help of the following auxiliary result.

Proposition 4.3. *The formulae*

$$(v,\mu_1)\mapsto \int_{-\infty}^{y} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}t, \qquad (v,\mu_1)\mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}s\,\mathrm{d}t$$

define analytic mapppings $\mathcal{U}\times\Lambda_1 \to L^2(-\infty,0)$ *and*

$$\int_{-\infty}^{0} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}y = [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\eta]_0]\!]_0 \gamma\sin\theta_2$$

for each $(v,\mu_1)\in\mathcal{U}\times\Lambda_1$.

Proof. It follows from (3.17) that

$$[\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 = [\![[(g_0^{\mu_1}(v))_y]_0]\!]_0,$$

where

$$\begin{aligned} g_0^{\mu_1}(v) &= \Phi_y - K_1(\eta)\eta\Phi_y - K_1(\eta)\Psi\rho - \frac{K_1(\eta)}{K_2(\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z \\ &\qquad - (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_1(\eta)\eta_z\Psi \\ &= \Phi_y - \Big(K_2(\eta)\eta\Phi_y - K_2(\eta)\Psi\rho - (\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(\Phi_z - K_1(\eta)\eta_z\Phi_y)\eta_z \\ &\qquad\qquad - (\nu_0+\mu_1)\cos(\theta_1-\theta_2)K_2(\eta)\eta_z\Psi\Big)\mathrm{e}^y \end{aligned}$$

and $v=(\eta,\rho,\Phi,\zeta,\xi,\Psi)$. Observing that $(v,\mu_1)\mapsto g_0^{\mu_1}(v)$ maps $U\times\Lambda_1$ analytically into $L^2(-\infty,0)$ (see Proposition 3.1) and the same is true of $(v,\mu_1)\mapsto \int_{-\infty}^{y} g_0^{\mu_1}(v)\,\mathrm{d}t$ because $u\mapsto \int_{-\infty}^{y} u(t)\mathrm{e}^t\,\mathrm{d}t$ belongs to $\mathcal{L}(L^2(-\infty,0))$, we conclude that $(v,\mu_1)\mapsto \int_{-\infty}^{y} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}t$ and $(v,\mu_1)\mapsto \int_{-\infty}^{y}\int_{-\infty}^{t} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}s\,\mathrm{d}t$ map $\mathcal{U}\times\Lambda_1$ analytically into $L^2(-\infty,0)$. Finally

$$\int_{-\infty}^{0} [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0 \,\mathrm{d}y = [\![[\rho]_0]\!]_0\gamma\sin\theta_2 = [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\eta]_0]\!]_0 \gamma\sin\theta_2$$

because of (3.18), (3.19) and (3.11). □

Lemma 4.4. *The formulae* (4.13) *define analytic mappings* $\mathcal{U}\times\Lambda_1\times\Lambda_2 \to L^2(-\infty,0)$ *and the formula* (4.14) *is satisfied for each* $(\hat{v},\mu_1,\mu_2)\in\mathcal{U}\times\Lambda_1\times\Lambda_2$.

Proof. We first note that

$$\widetilde{\mathrm{d}G}^{\mu_1}[v]^*(\hat{w}) = J\big((\kappa_0+\mu_2)v_\tau - v_{\mathrm{H}}^{\mu_1}(v)\big), \tag{4.15}$$

where $v = K^{\mu_1}(\hat{v})$, and using the explicit formulae for $\widetilde{\mathrm{d}G}^{\mu_1}[v]^*$ appearing in the proof of Proposition 3.5, we find that the ζ- and Γ-components of equation (4.15) are

$$\hat{w}_\zeta = (\kappa_0+\mu_2)\frac{\mathrm{d}}{\mathrm{d}\tau}v_\eta - \left(v_{\mathrm{H}}^{\mu_1}(w)\right)_\eta, \tag{4.16}$$

$$\hat{w}_\Gamma - (g_1^{\mu_1}(v,\hat{w}))_y - (g_2(v,\hat{w}))_z = -(\kappa_0+\mu_2)\frac{\mathrm{d}}{\mathrm{d}\tau}v_\Psi + (v_{\mathrm{H}}^{\mu_1}(v))_\Psi\,, \tag{4.17}$$

where

$$\begin{aligned} g_1^{\mu_1}(v,\hat{w}) &= \left(K_2(v_\eta)K_3 v_\eta - \frac{K_1^2(v_\eta)}{K_2(v_\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z^2\right)\Delta^{-1}((\hat{w}_\Gamma)_y) \\ &= \left(K_2(v_\eta)v_\eta - K_1(v_\eta)(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z^2\right)\mathrm{e}^y\Delta^{-1}((\hat{w}_\Gamma)_y), \\ g_2(v,\hat{w}) &= \left(\frac{K_1(v_\eta)}{K_2(v_\eta)}(\nu_0+\mu_1)^2\sin^2(\theta_1-\theta_2)(v_\eta)_z\right)\Delta^{-1}((\hat{w}_\Gamma)_y); \end{aligned}$$

note in particular that $(\hat{v},\mu_1)\mapsto g_1^{\mu_1}(v,\hat{w})$ maps $\hat{U}\times\Lambda_1$ analytically into $L^2(-\infty,0)$ (see Proposition 3.1) and the same is true of $(\hat{v},\mu_1)\mapsto\int_{-\infty}^y g_1^{\mu_1}(v,\hat{w})\,\mathrm{d}t$ because $u\mapsto\int_{-\infty}^y u(t)\mathrm{e}^t\,\mathrm{d}t$ belongs to $\mathcal{L}(L^2(-\infty,0))$.

Equation (4.17) implies that

$$[\![[\hat{w}_\Gamma]_0]\!]_0 = [\![[(g_1^{\mu_1}(v,\hat{w}))_y]_0]\!]_0 + [\![[(v_{\mathrm{H}}^{\mu_1}(v))_\Psi]_0]\!]_0,$$

and it follows from this identity and Proposition 4.3 that $(\hat{v},\mu_1,\mu_2)\mapsto\int_{-\infty}^y[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}t$ and $(\hat{v},\mu_1,\mu_2)\mapsto\int_{-\infty}^y\int_{-\infty}^t[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}s\,\mathrm{d}t$ map $\mathcal{U}\times\Lambda_1\times\Lambda_2$ analytically into $L^2(-\infty,0)$. Furthermore, the calculation

$$\begin{aligned} \int_{-\infty}^0[\![[\hat{w}_\Gamma]_0]\!]_0\,\mathrm{d}y &= \underbrace{\left[[\![[g_1^{\mu_1}(w,\hat{w})]_0]\!]_0\right]_{-\infty}^0}_{=0} + \int_{-\infty}^0[\![[(v_{\mathrm{H}}^{\mu_1}(w))_\Psi]_0]\!]_0\,\mathrm{d}y \\ &= [\![(v_{\mathrm{H}}^{\mu_1}(w))_\eta]_0]\!]_0\gamma\sin\theta_2 \\ &= -[\![[\hat{w}_\zeta]_0]\!]_0\gamma\sin\theta_2, \end{aligned}$$

shows that (4.14) is also satisfied; here we have used Proposition 4.3 and the facts that

$$[\![[\hat{w}_\zeta]_0]\!]_0 = -[\![[(v_{\mathrm{H}}^{\mu_1}(w))_\eta]_0]\!]_0$$

(see equation (4.16)) and $\Delta^{-1}((\hat{w}_\Gamma)_y)\big|_{y=0}=0$. □

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