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A plethora of fully localised solitary waves for the full-dispersion Kadomtsev–Petviashvili equation

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Abstract

The KP-I equation arises as a weakly nonlinear model equation for gravity-capillary waves with Bond number $\beta > 1/3$, also called strong surface tension. This equation has recently been shown to have a family of nondegenerate, symmetric 'fully localised' or 'lump' solitary waves which decay to zero in all spatial directions. The full-dispersion KP-I equation is obtained by retaining the exact dispersion relation in the modelling from the water-wave problem. In this paper we show that the FDKP-I equation also has a family of symmetric fullly localised solitary waves which are obtained by casting it as a perturbation of the KP-I equation and applying a suitable variant of the implicit-function theorem.

1. Introduction

1.1. The KP and FDKP equations

The full-dispersion Kadomtsev–Petviashvili (FDKP) equation

$$u_t + m(\mathbf{D})u_x + 2uu_x = 0, \tag{1.1}$$

where the Fourier multiplier m is given by

$$m(\mathbf{D}) = \left(1 + \beta|\mathbf{D}|^2\right)^{\frac{1}{2}} \left(\frac{\tanh |\mathbf{D}|}{|\mathbf{D}|}\right)^{\frac{1}{2}} \left(1 + \frac{2D_2^2}{D_1^2}\right)^{\frac{1}{2}}$$

with $\mathbf{D} = -i(\partial_x, \partial_y)$, was introduced by Lannes (2013) (see also Lannes and Saut (2014)) as an alternative to the classical Kadomtsev–Petviashvili (KP) equation

$$(\zeta_t - 2\zeta\zeta_x + \tfrac{1}{2}(\beta - \tfrac{1}{3})\zeta_{xxx})_x - \zeta_{yy} = 0, \tag{1.2}$$

which arises as a weakly nonlinear approximation for three-dimensional gravity-capillary water waves. The parameter $\beta > 0$ measures the relative strength of surface tension; the case $\beta > \frac{1}{3}$ for strong surface tension is termed KP-I, while the case $\beta < \frac{1}{3}$ for weak surface tension is termed KP-II. The analogous convention is used for the full-dispersion FDKP equation, giving us an FDKP-I equation for the strong surface tension case studied in this paper.

An FDKP solitary wave is a nontrivial, evanescent solution of (1.1) of the form $u(x, y, t) = u(x - ct, y)$ with wave speed $c > 0$, that is, a localised solution of the equation

$$- cu + m(\mathbf{D})u + u^2 = 0. \tag{1.3}$$

Similarly, a KP solitary wave is a nontrivial, evanescent solution of (1.2) of the form $\zeta(x, y, t) = \zeta(x - \tilde{c}t, y)$ with wave speed $\tilde{c} > 0$, that is, a localised solution of the equation

$$(\tilde{c} - 1)\zeta + \tilde{m}(\mathbf{D})\zeta + \zeta^2 = 0, \tag{1.4}$$

where

$$\tilde{m}(\mathbf{D}) = 1 + \frac{D_2^2}{D_1^2} + \tfrac{1}{2}(\beta - \tfrac{1}{3})D_1^2.$$

Let us emphasise that these waves are fully localised, that is, decaying in all spatial directions. The KP equation allows a scaling, such that the wave speed $\tilde{c}$ can be normalised to unity by the transformation $\zeta(x, y) \mapsto \tilde{c}\zeta(\tilde{c}^{\frac{1}{2}}x, \tilde{c}y)$, which converts (1.4) into the equation

$$\tilde{m}(\mathbf{D})\zeta + \zeta^2 = 0. \tag{1.5}$$

While it is known that the KP-II equation does not admit any solitary waves (de Bouard and Saut 1997), the situation is rather different for the KP-I equation. Letting $\zeta(x, y) = \zeta(\tilde{x}, \tilde{y})$ with $(\tilde{x}, \tilde{y}) = (\tfrac{1}{2}(\beta - \tfrac{1}{3}))^{\frac{1}{2}}(x, y)$, one can write the KP-I equation in the alternative form

$$\partial_x^2(-\partial_x^2\zeta + \zeta + \zeta^2) + \partial_y^2\zeta = 0, \tag{1.6}$$

in which we have dropped the tildes for notational simplicity. This equation has a family of explicit symmetric 'lump' solutions of the form

$$\zeta_k^\star(x, y) = -6\partial_x^2 \log \tau_k^\star(x, y), \qquad k = 1, 2, \dots, \tag{1.7}$$

where $\tau_k^\star$ is a polynomial of degree $k(k + 1)$ with $\tau_k^\star(x, y) = \tau_k^\star(-x, y) = \tau_k^\star(x, -y)$ for all $(x, y) \in \mathbb{R}^2$; the first two members of the family are

$$\tau_1^\star(x, y) = x^2 + y^2 + 3,$$
$$\tau_2^\star(x, y) = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 25x^4 + 90x^2y^2 + 17y^4 - 125x^2 + 475y^2 + 1875.$$

Note that the KP lump solutions $\zeta_k^\star$ are smooth, decaying rational functions, so that the same is true of their derivatives of all orders. The functions $\zeta_1^\star$ and $\zeta_2^\star$ are sketched in Figure 1.

The following result was established by Liu and Wei (2019) and Liu, Wei and Yang (2024a, 2024b); see also the comments below the lemma.

Lemma 1.1.

  (i) *Every smooth, algebraically decaying solution of the KPI equation* (1.6) *has the form $\zeta(x, y) = -6\partial_x^2 \log \tau(x, y)$, for some polynomial τ of degree $k(k + 1)$ with $k \in \mathbb{N}$ and satisfies $|\zeta(x, y)| \lesssim (1 + x^2 + y^2)^{-1}$ for all $(x, y) \in \mathbb{R}^2$.*
 (ii) *There is a unique symmetric solution $\zeta_k^\star$ of the form* (1.7) *for each $k \in \mathbb{N}$ with $k(k + 1) \leq 600$.*
(iii) *The solutions $\zeta_1^\star$, $\zeta_2^\star$ are nondegenerate: the only smooth evanescent solution of the linearised equation*

$$\partial_x^2(-\partial_x^2 w + w + 2\zeta_k^\star w) + \partial_y^2 w = 0$$

for $k = 1, 2$ that is also invariant under $w(x, y) \mapsto w(-x, y)$ and $w(x, y) \mapsto w(x, -y)$ is $w(x, y) = 0$.
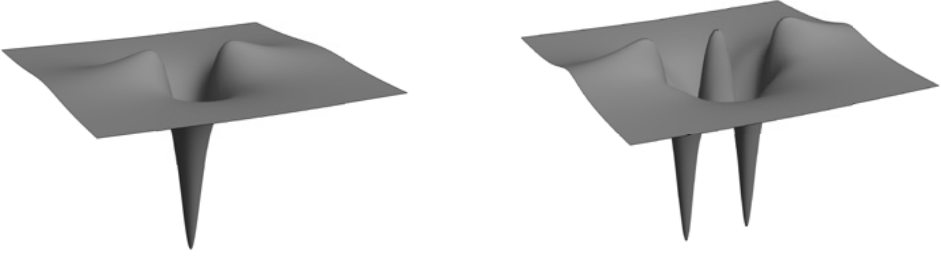
Figure 1. *The KP lumps $\zeta_1^\star$ (left) and $\zeta_2^\star$ (right).*

It is conjectured that part (ii) actually holds for all $k \in \mathbb{N}$ (Liu, Wei and Yang 2024b); furthermore a sketch of the proof of the nondegeneracy of $\zeta_k^\star$ for $k \geq 3$ was given by Liu, Wei and Yang (2024a), and here we accept the validity of this result. The existence of a solitary-wave solution to the FDKP-I equation was proved in Ehrnström and Groves (2018) using a variational method, and in this paper we considerably improve our previous result by using a perturbation argument in place of constrained minimisation to prove the following theorem, which establishes the existence of FDKP solitary waves 'close' to $\zeta_k^\star$ for all k for which (iii) holds.

Theorem 1.2. *For each $k \in \mathbb{N}$ and each sufficiently small value of $\varepsilon > 0$ the FDKP-I equation* (1.3) *possesses a smooth fully localised solitary-wave solution $u_k^\star$ of wave speed $c = 1 - \varepsilon^2$, which satisfies*

$$u_k^\star(x, y) = u_k^\star(-x, y) = u_k^\star(x, -y)$$

for all $(x, y) \in \mathbb{R}^2$ and

$$u_k^\star(x, y) = \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + o(\varepsilon^2) \tag{1.8}$$

uniformly over $(x, y) \in \mathbb{R}^2$.

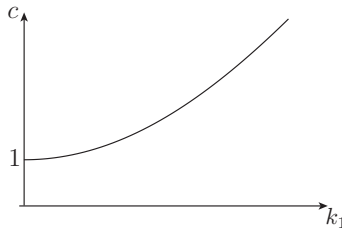Theorem 1.2 is proved in Sections 2–4 below.

1.2. The method

Figure 2. *FKDP-I dispersion relation for two-dimensional wave trains.*

To motivate our method it is instructive to review the formal derivation of the steady KP equation (1.5) from the steady FDKP equation (1.3). We begin with the linear dispersion relation for the time-dependent FDKP equation (1.1) with $\beta > \frac{1}{3}$: the speed c and wave number k_1 of a two-dimensional sinusoidal travelling wave train satisfy

$$c = \left(1 + \beta |k_1|^2\right)^{\frac{1}{2}} \left(\frac{\tanh |k_1|}{|k_1|}\right)^{\frac{1}{2}}.$$

The function $k_1 \mapsto c(k_1)$, $k_1 \geq 0$ has a unique global minimum at $k_1 = 0$ with $c(0) = 1$ (see Figure 2). Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that $c'(k_1) = 0$ (see Dias and Kharif (1999, §3)), and one therefore expects bifurcation of small-amplitude solitary waves from uniform flow with unit speed. Furthermore, observing that m is an analytic function of k_1 and $\frac{k_2}{k_1}$ (note that $|\mathbf{k}|^2 = k_1^2 + \frac{k_2^2}{k_1^2} k_1^2$ for $\mathbf{k} = (k_1, k_2)$), one finds that

$$m(\mathbf{k}) = \tilde{m}(\mathbf{k}) + O(|(k_1, \tfrac{k_2}{k_1})|^4) \tag{1.9}$$

as $(k_1, \frac{k_2}{k_1}) \to 0$. The variables $(k_1, \frac{k_2}{k_1})$ are intrinsic to the steady KP equation (1.5), and corresponding to them is the scaling

$$u(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y). \tag{1.10}$$

Substituting the Ansatz (1.10) with assumed wave speed

$$c = 1 - \varepsilon^2$$

into the steady FDKP equation (1.3), one finds that, to leading order, ζ also satisfies the normalised KP equation (1.5). We henceforth assume that $c = 1 - \varepsilon^2$ for $0 < \varepsilon < \varepsilon_0$, where ε_0 is taken small enough for all our arguments to be valid.
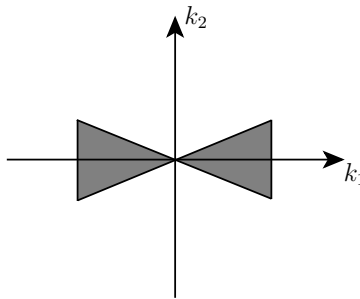

Figure 3. *The cone $C = \{\mathbf{k} \in \mathbb{R}^2 : |k_1| \leq \delta, |\frac{k_2}{k_1}| \leq \delta\}$.*

In the rigorous theory we seek solutions of (1.3) in a suitable function space X and identify a corresponding phase space Z for this equation. These spaces, which are defined precisely in Section 2, satisfy $X \subseteq Z \subseteq L^2(\mathbb{R}^2)$. The relationship (1.9) between the symbols m and $\tilde{m}$ suggests that the spectrum of a solitary wave u is concentrated in the region $|k_1|, |\frac{k_2}{k_1}| \ll 1$. We therefore decompose $L^2(\mathbb{R}^2)$, and hence also X and Z, into the direct sum of subspaces of functions whose spectra are supported in the region

$$C = \left\{\mathbf{k} \in \mathbb{R}^2 : |k_1| \leq \delta, \left|\frac{k_2}{k_1}\right| \leq \delta\right\} \tag{1.11}$$

and its complement, where δ is a small but fixed positive number (see Figure 3), so that

$$X = \underbrace{\chi(\mathbf{D})X}_{= X_1} \oplus \underbrace{(1 - \chi(\mathbf{D}))X}_{= X_2}, \qquad Z = \underbrace{\chi(\mathbf{D})Z}_{= Z_1} \oplus \underbrace{(1 - \chi(\mathbf{D}))Z}_{= Z_2},$$

in which χ is the characteristic function of C. Observing that X_1, Z_1 both coincide with $\chi(\mathbf{D})L^2(\mathbb{R}^2)$, we equip Z_1 with the $L^2(\mathbb{R}^2)$ norm and X_1 with the equivalent scaled norm

$$|u_1|_\varepsilon^2 = \int_{\mathbb{R}^2} \left(|u|^2 + \varepsilon^2 |D_1 u|^2 + \varepsilon^2 \left| \frac{D_2}{D_1} u \right|^2 \right) dx\,dy, \tag{1.12}$$

and employ a method akin to the Lyapunov–Schmidt reduction to determine $u_2 \in X_2$ as a function of $u_1 \in X_1$. With $n = m - 1$, the result is the equation

$$\varepsilon^2 u_1 + n(\mathbf{D})u_1 + \chi(\mathbf{D})(u_1 + u_2(u_1))^2 = 0,$$

for u_1 in the unit ball

$$U = \{ u_1 \in X_1 \colon |u_1|_\varepsilon \leq 1 \},$$

of X_1.

Applying the KP scaling

$$u_1(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y)$$

so that the spectrum of ζ lies in the set

$$C_\varepsilon = \left\{ \mathbf{k} \in \mathbb{R}^2 \colon |k_1| \leq \frac{\delta}{\varepsilon}, \left| \frac{k_2}{k_1} \right| \leq \frac{\delta}{\varepsilon} \right\},$$

one obtains the reduced equation

$$\varepsilon^{-2} n_\varepsilon(\mathbf{D}) \zeta + \zeta + \chi_\varepsilon(\mathbf{D}) \zeta^2 + S_\varepsilon(\zeta) = 0, \tag{1.13}$$

where

$$n_\varepsilon(\mathbf{k}) = n(\varepsilon k_1, \varepsilon^2 k_2), \qquad \chi_\varepsilon(\mathbf{k}) = \chi(\varepsilon k_1, \varepsilon^2 k_2).$$

The remainder term $S_\varepsilon \colon B_M(0) \subseteq \chi_\varepsilon(\mathbf{D})Y^1 \to \chi_\varepsilon(\mathbf{D})L^2(\mathbb{R}^2)$ satisfies the estimates

$$|S_\varepsilon(\zeta)|_{L^2} \lesssim \varepsilon^2 |\zeta|_{Y^1}^3, \quad |dS_\varepsilon[\zeta]|_{\mathcal{L}(Y^1, L^2(\mathbb{R}^2))} \lesssim \varepsilon^2 |\zeta|_{Y^1}^2$$

(see Section 3), where

$$Y^1 = \{ u \in L^2(\mathbb{R}^2) \colon |u|_{Y^1} < \infty \}, \qquad |u|_{Y^1}^2 = \int_{\mathbb{R}^2} \left(|u|^2 + |D_1 u|^2 + \left| \frac{D_2}{D_1} u \right|^2 \right) dx\,dy$$

is the natural energy space for the KP-I equation (de Bouard and Saut 1997); the constant $M > 1$ is chosen large enough so that $\zeta_k^\star \in B_M(0)$, while the requirement that $B_M(0)$ is contained in the range of the isomorphism $u_1 \mapsto \zeta$ requires $\varepsilon \lesssim M^{-2}$. In the formal limit $\varepsilon \to 0$ the subspace $\chi_\varepsilon(\mathbf{D})Y^1$ 'fills out' all of Y^1 and equation (1.13) reduces to the KP equation (1.5).

We demonstrate in Theorem 4.2 that equation (1.13) has solutions ζ_k^ε which satisfy $\zeta_k^\varepsilon \to \zeta_k^\star$ as $\varepsilon \to 0$ in a suitable subspace of Y^1, and deduce our main Theorem 1.2 by tracing back the steps in the reduction procedure. One of the key arguments is based upon the nondegeneracy result given in Lemma 1.1(iii), which allows one to apply a variant of the implicit-function theorem. For this purpose we exploit the fact that the reduction procedure preserves the invariance of equation (1.3) under $u(x, y) \mapsto u(-x, y)$ and $u(x, y) \mapsto u(x, -y)$, so that equation (1.13) is invariant under $\zeta(x, y) \mapsto \zeta(-x, y)$ and $\zeta(x, y) \mapsto \zeta(x, -y)$. It is necessary to use a low regularity version of the implicit-function theorem since the reduction in Section 3 is performed using the ε-dependent norm $|\cdot|_\varepsilon$ and thus does not yield information concerning the smoothness of u_1 as a function of ε.

Ehrnström and Groves (2018) use a variational version of the reduction procedure outlined above to reduce a variational principle for equation (1.3) to a variational principle for (1.13) and proceed by finding critical points of the reduced variational functional by the direct methods of the calculus of variations. Here, with some amendments, we use their functional-analytic setting and follow the steps in their reduction (see Sections 2 and 3 below), but study the reduced equation (1.13) in Section 4 in an entirely different manner, arriving at a much more comprehensive conclusion. Note that the perturbation argument in Section 4 has also been applied to the Whitham equation by Stefanov and Wright (2020) and moreover to the full water-wave problem, namely by Groves (2021) for two-dimensional gravity-capillary solitary water waves and by Buffoni et al (2022) for three-dimensional gravity-capillary solitary waves on water of infinite depth.

2. Function spaces

In this section we introduce the Banach spaces used in our theory and state their main properties; the proofs of most of these results are given by Ehrnström and Groves (2018, §2). We use the familiar scale $\{H^r(\mathbb{R}^2), |\cdot|_{H^r}\}_{r \geq 0}$ of Sobolev spaces together with the anisotropic spaces

$$X = \{u \in L^2(\mathbb{R}^2) \colon |u|_X < \infty\}, \qquad |u|_X^2 = \int_{\mathbb{R}^2} \left(1 + \frac{k_2^2}{k_1^2} + \frac{k_2^4}{k_1^2} + |\mathbf{k}|^{2s}\right) |\hat{u}(\mathbf{k})|^2 \, d\mathbf{k},$$

$$Z = \{u \in L^2(\mathbb{R}^2) \colon |u|_Z < \infty\}, \qquad |u|_Z^2 = \int_{\mathbb{R}^2} \left(1 + |\mathbf{k}| + k_1^2 |\mathbf{k}|^{2s-3}\right) |\hat{u}(\mathbf{k})|^2 \, d\mathbf{k},$$

in which the Sobolev index $s > \frac{3}{2}$ is fixed and $u \mapsto \hat{u}$ denotes the unitary Fourier transform on $L^2(\mathbb{R}^2)$. We also use the scale $\{Y^r, |\cdot|_{Y^r}\}_{r \geq 0}$, where

$$Y^r = \{u \in L^2(\mathbb{R}^2) \colon |u|_{Y^r} < \infty\}, \qquad |u|_{Y^r}^2 = \int_{\mathbb{R}^2} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^r |\hat{u}(\mathbf{k})|^2 \, d\mathbf{k}. \qquad (2.1)$$

Note that $Y^0 = L^2(\mathbb{R}^2)$ while Y^1 is the natural energy space for the KP-I equation (de Bouard and Saut 1997). Ehrnström and Groves (2018) use only the space Y^1, there called $\tilde{Y}$, but the proof of the following proposition is a straightforward variant of the proof of Lemma 2.1(i) in that reference.

Proposition 2.1. One has the continuous embeddings

$$X \hookrightarrow Y^r \hookrightarrow L^2(\mathbb{R}^2), \quad H^{s-\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow Z \hookrightarrow L^2(\mathbb{R}^2), \quad X \hookrightarrow H^s(\mathbb{R}^2)$$

for all $r \in [0, s]$, and in particular $X \hookrightarrow C_b(\mathbb{R}^2)$, the space of bounded, continuous functions on $\mathbb{R}^2$.

Proposition 2.2. The space Y^1 (and hence Y^r for each $r \geq 1$) is

(i) continuously embedded in $L^p(\mathbb{R}^2)$ for $2 \leq p \leq 6$,
(ii) compactly embedded in $L^p_{\text{loc}}(\mathbb{R}^2)$ for $2 \leq p < 6$.

Proposition 2.3. The space Y^r is continuously embedded in $C_b(\mathbb{R}^2)$ for each $r > \frac{3}{2}$.

Proof. Note that

$$|u|_\infty \lesssim \int_{\mathbb{R}^2} |\hat{u}(\mathbf{k})| \, d\mathbf{k} = \int_{\mathbb{R}^2} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{-\frac{1}{2}r} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}r} |\hat{u}(\mathbf{k})| \, d\mathbf{k} \leq |u|_{Y^r} I^{\frac{1}{2}},$$

where, with a change of variables,

$$I = \int_{\mathbb{R}^2} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{-r} \mathrm{d}\mathbf{k} = \int_{\mathbb{R}^2} (1 + |\mathbf{k}|^2)^{-r} k_1 \, \mathrm{d}\mathbf{k} < \infty.$$

The continuity of u follows from a standard dominated convergence argument with $\hat{u}$ as dominating function. □

Proposition 2.4.

(i) The Fourier multiplier $m(\mathbf{D})$ maps X continuously onto Z.
(ii) The formula $u \mapsto u^2$ maps X smoothly into Z.

We decompose $u \in L^2(\mathbb{R}^2)$ into the sum of functions u_1 and u_2 whose spectra are supported in the region C defined in (1.11) and its complement (see Figure 3) by writing

$$u_1 = \chi(\mathbf{D})u, \qquad u_2 = (1 - \chi(\mathbf{D}))u,$$

where χ is the characteristic function of C. Since they are subspaces of $L^2(\mathbb{R}^2)$, the Fourier multiplier $\chi(\mathbf{D})$ induces the orthogonal decomposition $X = X_1 \oplus X_2$ with $X_1 = \chi(\mathbf{D})X$, $X_2 = (1 - \chi(\mathbf{D}))X$ and analogous decompositions of the spaces Y^r and Z. We write $Z = Z_1 \oplus Z_2$, but retain the explicit notation $\chi(\mathbf{D})Y^r$ and $\chi(\mathbf{D})L^2(\mathbb{R}^2)$. The spaces X_1, Z_1 and $\chi(\mathbf{D})Y^r$ all coincide with $\chi(\mathbf{D})L^2(\mathbb{R}^2)$, and $|\cdot|_{L^2}$, $|\cdot|_X$, $|\cdot|_{Y^r}$ and $|\cdot|_Z$ are all equivalent norms for these spaces. We do however make specific choices in the theory in below; we equip Z_1 and $\chi(\mathbf{D})Y^r$ with $|\cdot|_{L^2}$ and $|\cdot|_{Y^r}$ respectively, and X_1 with the equivalent scaled norm

$$|u_1|_\varepsilon^2 = \int_{\mathbb{R}^2} \left(1 + \varepsilon^{-2}k_1^2 + \varepsilon^{-2}\frac{k_2^2}{k_1^2}\right)|\hat{u}_1(\mathbf{k})|^2 \, \mathrm{d}\mathbf{k}$$

(see equation (1.12)) in anticipation of the KP scaling $(k_1, k_2) \mapsto (\varepsilon k_1, \varepsilon^2 k_2)$.

Proposition 2.5. The mapping $n(\mathbf{D})$, where $n = m - 1$, is an isomorphism $X_2 \to Z_2$.

Proposition 2.6. The estimates

$$|\partial_x^{m_1} \partial_y^{m_2} u_1|_\infty \lesssim \varepsilon |u_1|_\varepsilon, \qquad m_1, m_2 \geq 0,$$

and

$$|u_1 v|_Z \lesssim \varepsilon |u_1|_\varepsilon |v|_X, \qquad |vw|_Z \lesssim |v|_X |w|_X$$

hold for all $u_1 \in X_1$ and $v, w \in X$.

Finally, we introduce the space $Y_\varepsilon^r = \chi_\varepsilon(\mathbf{D})Y^r$, where $\chi_\varepsilon(k_1, k_2) = \chi(\varepsilon k_1, \varepsilon^2 k_2)$ (with norm $|\cdot|_{Y^r}$), noting the relationship

$$|u|_\varepsilon^2 = \varepsilon |\zeta|_{Y^1}^2, \qquad u(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y)$$

for $\zeta \in Y_\varepsilon^1$. Observe that Y_ε^r coincides with $\chi_\varepsilon(\mathbf{D})X$, $\chi_\varepsilon(\mathbf{D})Z$ and $\chi_\varepsilon(\mathbf{D})L^2(\mathbb{R}^2)$ for $\varepsilon > 0$, and with $\chi(\mathbf{D})Y^r$ in the limit $\varepsilon \to 0$.

3. Reduction

We proceed by making the Ansatz $c = 1 - \varepsilon^2$ and studying equation (1.3) in its phase space Z. Note that $u = u_1 + u_2 \in X_1 \oplus X_2$ satisfies this equation if and only if

$$\varepsilon^2 u_1 + n(\mathbf{D})u_1 + \chi(\mathbf{D})(u_1 + u_2)^2 = 0, \qquad \text{in } Z_1, \tag{3.1}$$

$$\varepsilon^2 u_2 + n(\mathbf{D})u_2 + (1 - \chi(\mathbf{D}))(u_1 + u_2)^2 = 0, \qquad \text{in } Z_2. \tag{3.2}$$

The first step is to solve (3.2) for u_2 as a function of u_1 using the following result, which is proved by a straightforward application of the contraction mapping principle.

Theorem 3.1. *Let $\mathcal{W}_1$, $\mathcal{W}_2$ be Banach spaces, K be a continuous function $\overline{B}_1(0) \subseteq \mathcal{W}_1 \to [0, \infty)$ and $\mathcal{F} \colon \overline{B}_1(0) \times \mathcal{W}_2 \to \mathcal{W}_2$ be a smooth function satisfying*

$$|\mathcal{F}(w_1, 0)|_{\mathcal{W}_2} \leq \tfrac{1}{2} K(w_1), \qquad |d_2 \mathcal{F}[w_1, w_2]|_{\mathcal{L}(\mathcal{W}_2, \mathcal{W}_2)} \leq \tfrac{1}{3}$$

for all $(w_1, w_2) \in \overline{B}_1(0) \times \overline{B}_{K(w_1)}(0)$. The fixed-point equation

$$w_2 = \mathcal{F}(w_1, w_2)$$

has for each $w_1 \in \overline{B}_1(0)$ a unique solution $w_2 = w_2(w_1) \in \overline{B}_{K(w_1)}(0)$. Moreover w_2 is a smooth function of w_1 and satisfies

$$|dw_2[w_1]|_{\mathcal{L}(\mathcal{W}_1, \mathcal{W}_2)} \lesssim |d_1 \mathcal{F}[w_1, w_2(w_1)]|_{\mathcal{L}(\mathcal{W}_1, \mathcal{W}_2)}.$$

Write (3.2) as

$$u_2 = \mathcal{F}(u_1, u_2), \tag{3.3}$$

where

$$\mathcal{F}(u_1, u_2) = -n(\mathbf{D})^{-1}(1 - \chi(\mathbf{D})) \left(\varepsilon^2 u_2 + (u_1 + u_2)^2 \right); \tag{3.4}$$

the following mapping property of $\mathcal{F}$ follows from Propositions 2.4(ii) and 2.5.

Proposition 3.2. Equation (3.4) defines a smooth mapping $\mathcal{F} \colon X_1 \times X_2 \to X_2$.

Lemma 3.3. *Define $U = \{u_1 \in X_1 \colon |u_1|_\varepsilon \leq 1\}$. Equation (3.3) defines a map*

$$U \ni u_1 \mapsto u_2(u_1) \in X_2$$

which satisfies

$$|u_2(u_1)|_{X_2} \lesssim \varepsilon |u_1|_\varepsilon^2, \qquad |du_2[u_1]|_{\mathcal{L}(X_1, X_2)} \lesssim \varepsilon |u_1|_\varepsilon.$$

Proof. We apply Theorem 3.1 to equation (3.3) with $\mathcal{W}_1 = (X_1, |\cdot|_\varepsilon)$, $\mathcal{W}_2 = (X_2, |\cdot|_X)$. Note that

$$d_1 \mathcal{F}[u_1, u_2](v_1) = -n(\mathbf{D})^{-1}(1 - \chi(\mathbf{D}))(2(u_1 + u_2)v_1),$$
$$d_2 \mathcal{F}[u_1, u_2](v_2) = -n(\mathbf{D})^{-1}(1 - \chi(\mathbf{D}))(\varepsilon^2 v_2 + 2(u_1 + u_2)v_2)$$

and

$$|(n(\mathbf{D}))^{-1}(1 - \chi(\mathbf{D}))u|_X \lesssim |u|_Z,$$

by Proposition 2.5. Using Proposition 2.6, we therefore find that

$$|\mathcal{F}(u_1, 0)|_X = |u_1^2|_Z \lesssim \varepsilon |u_1|_\varepsilon |u_1|_X \lesssim \varepsilon |u_1|_\varepsilon |u_1|_{L^2} \leq \varepsilon |u_1|_\varepsilon^2$$

and

$$|d_2 \mathcal{F}[u_1, u_2](v_2)|_X \lesssim \varepsilon^2 |v_2|_Z + |u_1 v_2|_Z + |u_2 v_2|_Z$$
$$\lesssim (\varepsilon^2 + \varepsilon|u_1|_\varepsilon + |u_2|_X)|v_2|_X.$$

To satisfy the assumptions of Theorem 3.1, we choose $K(u_1) = \sigma\varepsilon|u_1|_\varepsilon^2$ for a sufficiently large value of $\sigma > 0$, so that

$$|u_2|_X \lesssim \tfrac{1}{2} K(u_1), \qquad |d_2\mathcal{F}[u_1, u_2]|_{\mathcal{L}(X_2, X_2)} \lesssim \varepsilon$$

for $(u_1, u_2) \in U \times \overline{B}_{K(u_1)}(0)$. The lemma asserts the existence of a unique solution $u_2(u_1) \in \overline{B}_{K(u_1)}(0)$ of (3.3) for each $u_1 \in U$ which satisfies

$$|u_2(u_1)|_X \lesssim \varepsilon|u_1|_\varepsilon^2$$

and

$$|du_2[u_1](v_1)|_X \lesssim |d_1\mathcal{F}[u_1, u_2(u_1)](v_1)|_X$$
$$\lesssim |u_1 v_1|_Z + |u_2(u_1) v_1|_Z$$
$$\lesssim \varepsilon(|u_1|_X + |u_2(u_1)|_X)|v_1|_\varepsilon$$
$$\lesssim \varepsilon(|u_1|_\varepsilon + \varepsilon|u_1|_\varepsilon^2)|v_1|_\varepsilon,$$

where we have used Proposition 2.6. $\qquad\qquad\square$

Our next result shows in particular that $u = u_1 + u_2(u_1)$ belongs to $H^\infty(\mathbb{R}^2) = \bigcap\limits_{j=1}^{\infty} H^j(\mathbb{R}^2)$ for each $u_1 \in U_1$.

Proposition 3.4. Any function $u = u_1 + u_2 \in X_1 \oplus X_2$ which satisfies (3.3) belongs to $H^\infty(\mathbb{R}^2)$.

Proof. Obviously $u_1 \in H^\infty(\mathbb{R}^2)$, and to show that u_2 is also smooth we abandon the fixed regularity index in the spaces X and Z and state it explicitly as a variable parameter. Since $H^s(\mathbb{R}^2)$ is an algebra for $s > \frac{3}{2}$ and $X_2^s \hookrightarrow (1 - \chi(D))H^s(\mathbb{R}^2) \hookrightarrow Z_2^{s+\frac{1}{2}}$ (see Proposition 2.1), the mapping

$$X_1 \oplus X_2^s \ni (u_1, u_2) \mapsto -(1 - \chi(\mathbf{D}))\left(\varepsilon^2 u_2 + (u_1 + u_2)^2\right) \in Z_2^{s+\frac{1}{2}}$$

is continuous. It follows that $u_2 \in X_2^{s+\frac{1}{2}}$, because $n(\mathbf{D})$ is an isomorphism $X_2^{s+\frac{1}{2}} \to Z_2^{s+\frac{1}{2}}$ by Proposition 2.5. Bootstrapping this argument yields $u_2 \in X_2^s \subset H^s(\mathbb{R}^2)$ for any $s \in \mathbb{R}$. $\qquad\square$

The next step is to substitute $u_2 = u_2(u_1)$ into equation (3.1) to obtain the reduced equation

$$\varepsilon^2 u_1 + n(\mathbf{D})u_1 + \chi(\mathbf{D})(u_1 + u_2(u_1))^2 = 0$$

for u_1. We can write this equation as

$$\varepsilon^2 u_1 + n(\mathbf{D})u_1 + \chi(\mathbf{D})u_1^2 + R_\varepsilon(u_1) = 0,$$

where

$$R_\varepsilon(u_1) = \chi(\mathbf{D})\left(2u_1 u_2(u_1) + u_2(u_1)^2\right). \tag{3.5}$$

Proposition 3.5. The function $R_\varepsilon \colon U \subseteq X_1 \to Z_1$ satisfies the estimates

$$|R_\varepsilon(u_1)|_{L^2} \lesssim \varepsilon^2 |u_1|_\varepsilon^3, \qquad |dR_\varepsilon[u_1]|_{\mathcal{L}(X_1, L^2(\mathbb{R}^2))} \lesssim \varepsilon^2 |u_1|_\varepsilon^2.$$

Proof. By using Proposition 2.6 and Lemma 3.3 it follows from (3.5) that

$$
\begin{aligned}
|R_\varepsilon(u_1)|_{L^2} &\lesssim |u_1 u_2(u_1)|_Z + |u_2(u_1)^2|_Z \\
&\lesssim \varepsilon |u_1|_\varepsilon |u_2(u_1)|_X + |u_2(u_1)|_X^2 \\
&\lesssim \varepsilon^2 |u_1|_\varepsilon^3,
\end{aligned}
$$

and from

$$
dR_\varepsilon[u_1](v_1) = \chi(\mathbf{D})\big(2v_1 u_2(u_1) + u_1 du_2[u_1](v_1) + 2u_2(u_1)du_2[u_1](v_1)\big)
$$

that

$$
\begin{aligned}
|dR_\varepsilon[u_1](v_1)|_{L^2} &\lesssim |v_1 u_2(u_1)|_Z + |u_1 du_2[u_1](v_1)|_Z + |u_2(u_1)du_2[u_1](v_1)|_Z \\
&\lesssim \varepsilon |v_1|_\varepsilon |u_2(u_1)|_X + \varepsilon |u_1|_\varepsilon |du_2[u_1](v_1))|_X + |u_2(u_1)|_X |du_2[u_1](v_1)|_X \\
&\lesssim \varepsilon^2 |u_1|_\varepsilon^2 |v_1|_\varepsilon. \qquad\qquad\qquad \square
\end{aligned}
$$

The reduction is completed by introducing the KP scaling

$$
u_1(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y),
$$

noting that $I: u_1 \mapsto \zeta$ is an isomorphism $X_1 \to Y_\varepsilon^1$ and $Z_1 \to Y_\varepsilon^0$ and choosing $M > 1$ large enough so that $\zeta_k^\star \in B_M(0)$ (and $\varepsilon > 0$ small enough so that $B_M(0) \subseteq Y_\varepsilon^1$ is contained in $I[U]$). Here we have replaced $(Z_1, |\cdot|_{L^2})$ by the identical space $(Y_\varepsilon^0, |\cdot|_{Y^0})$ in order to work exclusively with the scales $\{Y^r, |\cdot|_{Y^r}\}_{r \geq 0}$ and $\{Y_\varepsilon^r, |\cdot|_{Y^r}\}_{r \geq 0}$ of function spaces. We find that $\zeta \in B_M(0) \subseteq Y_\varepsilon^1$ satisfies the equation

$$
\varepsilon^{-2} n_\varepsilon(\mathbf{D})\zeta + \zeta + \chi_\varepsilon(\mathbf{D})\zeta^2 + S_\varepsilon(\zeta) = 0, \tag{3.6}
$$

which now holds in Y_ε^0, where

$$
n_\varepsilon(\mathbf{k}) = n(\varepsilon k_1, \varepsilon^2 k_2)
$$

and $S_\varepsilon: B_M(0) \subseteq Y_\varepsilon^1 \to Y_\varepsilon^0$ satisfies the estimates

$$
|S_\varepsilon(\zeta)|_{Y^0} \lesssim \varepsilon |\zeta|_{Y^1}^3, \quad |dS_\varepsilon[\zeta]|_{\mathcal{L}(Y^1, Y^0)} \lesssim \varepsilon |\zeta|_{Y^1}^2. \tag{3.7}
$$

Note that $|u_1|_\varepsilon^2 = \varepsilon |\zeta|_{Y^1}^2$ and that the change of variables from (x, y) to $(\varepsilon x, \varepsilon^2 y)$ introduces a further factor of $\varepsilon^{\frac{3}{2}}$ in the remainder term.

Finally, observe that the FDKP equation

$$
\varepsilon^2 u + n(\mathbf{D})u + u^2 = 0
$$

is invariant under $u(x, y) \mapsto u(-x, y)$ and $u(x, y) \mapsto u(x, -y)$ and the reduction procedure preserves this invariance: equation (3.6) is invariant under $\zeta(x, y) \mapsto \zeta(-x, y)$ and $\zeta(x, y) \mapsto \zeta(x, -y)$.

4. Solution of the reduced equation

In this section we construct solitary-wave solutions of the reduced equation (3.6), noting that in the formal limit $\varepsilon \to 0$ it reduces to the KP equation (1.5), which has explicit (symmetric) solitary-wave solutions $\zeta_k^\star$. For this purpose we use a perturbation argument, rewriting (3.6) as a fixed-point equation and applying the following variant of the implicit-function theorem. It is necessary to use a low regularity version of the implicit-function theorem since the reduction in Section 3 is performed using the ε-dependent norm $|\cdot|_\varepsilon$ and thus does not yield information concerning the smoothness of u_1 as a function of ε.

Theorem 4.1. *Let $\mathcal{W}$ be a Banach space, W_0 and Λ_0 be open neighbourhoods of respectively $w^\star$ in $\mathcal{W}$ and the origin in $\mathbb{R}$, and $\mathcal{G} \colon W_0 \times \Lambda_0 \to \mathcal{W}$ be a function which is differentiable with respect to $w \in W_0$ for each $\lambda \in \Lambda_0$. Furthermore, suppose that $\mathcal{G}(w^\star, 0) = 0$, $\mathrm{d}_1 \mathcal{G}[w^\star, 0] \colon \mathcal{W} \to \mathcal{W}$ is an isomorphism,*

$$\lim_{w \to w^\star} |\mathrm{d}_1 \mathcal{G}[w, 0] - \mathrm{d}_1 \mathcal{G}[w^\star, 0]|_{\mathcal{L}(\mathcal{W}, \mathcal{W})} = 0$$

and

$$\lim_{\lambda \to 0} |\mathcal{G}(w, \lambda) - \mathcal{G}(w, 0)|_{\mathcal{W}} = 0, \quad \lim_{\lambda \to 0} |\mathrm{d}_1 \mathcal{G}[w, \lambda] - \mathrm{d}_1 \mathcal{G}[w, 0]|_{\mathcal{L}(\mathcal{W}, \mathcal{W})} = 0$$

uniformly over $w \in W_0$.

There exist open neighbourhoods $W \subseteq W_0$ of $w^\star$ in $\mathcal{W}$ and $\Lambda \subseteq \Lambda_0$ of the origin in $\mathbb{R}$, and a uniquely determined mapping $h \colon \Lambda \to W$ with the properties that

(i) h is continuous at the origin with $h(0) = w^\star$,
(ii) $\mathcal{G}(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,
(iii) $w = h(\lambda)$ whenever $(w, \lambda) \in W \times \Lambda$ satisfies $\mathcal{G}(w, \lambda) = 0$.

Our main result is the following theorem, which is proved by reformulating equation (3.6) in an appropriately chosen function space and verifying that it satisfies the assumptions of Theorem 4.1 through a series of auxiliary results.

Theorem 4.2. *Fix $\theta \in (\frac{1}{2}, 1)$. For each sufficiently small value of $\varepsilon > 0$ equation (3.6) has a small-amplitude, symmetric solution ζ_k^ε in $Y_\varepsilon^{1+\theta}$ with $|\zeta_k^\varepsilon - \zeta_k^\star|_{Y^{1+\theta}} \to 0$ as $\varepsilon \to 0$.*

The first step in the proof of Theorem 4.2 is to write (3.6) as the fixed-point equation

$$\zeta + \varepsilon^2 \big(n_\varepsilon(\mathbf{D}) + \varepsilon^2\big)^{-1} \big(\chi_\varepsilon(\mathbf{D})\zeta^2 + S_\varepsilon(\zeta)\big) = 0 \tag{4.1}$$

and use the following results to 'replace' $\varepsilon^2 \big(n_\varepsilon(\mathbf{D}) + \varepsilon^2\big)^{-1}$ with $\tilde{m}(\mathbf{D})^{-1}$.

Proposition 4.3. The inequality

$$\left| \frac{\varepsilon^2}{\varepsilon^2 + n_\varepsilon(\mathbf{k})} - \frac{1}{\tilde{m}(\mathbf{k})} \right| \lesssim \frac{\varepsilon}{\big(1 + |(k_1, \frac{k_2}{k_1})|^2\big)^{1/2}}$$

holds uniformly over $|k_1|, |\frac{k_2}{k_1}| < \frac{\delta}{\varepsilon}$.

Proof. Recall that $\beta > \frac{1}{3}$. Clearly

$$\left| \frac{\varepsilon^2}{\varepsilon^2 + n_\varepsilon(\mathbf{k})} - \frac{1}{\tilde{m}(\mathbf{k})} \right| = \frac{\left| n_\varepsilon(\mathbf{k}) - (\beta - \frac{1}{3})\varepsilon^2 k_1^2 - \varepsilon^2 \frac{k_2^2}{k_1^2} \right|}{(\varepsilon^2 + n_\varepsilon(\mathbf{k})) \left(1 + (\beta - \frac{1}{3})k_1^2 + \frac{k_2^2}{k_1^2}\right)}.$$

Furthermore, since $n(s)$ is an analytic function of s_1 and $\frac{s_2}{s_1}$, we have that

$$\left| n(s) - \left(\beta - \frac{1}{3}\right)s_1^2 - \frac{s_2^2}{s_1^2} \right| \lesssim \left| \left(s_1, \frac{s_2}{s_1}\right) \right|^3$$

for $|(s_1, \frac{s_2}{s_1})| \le \delta$, and by the definition of n that

$$n(s) \gtrsim \left| \left(s_1, \frac{s_2}{s_1}\right) \right|^2$$

for all $(s_1, \frac{s_2}{s_1}) \in \mathbb{R}^2$. It follows that

$$\left| \frac{\varepsilon^2}{\varepsilon^2 + n_\varepsilon(\mathbf{k})} - \frac{1}{\tilde{m}(\mathbf{k})} \right| \lesssim \frac{\varepsilon |(k_1, \frac{k_2}{k_1})|^3}{(1 + |(k_1, \frac{k_2}{k_1})|^2)^2}$$

uniformly over $|k_1|, |\frac{k_2}{k_1}| < \frac{\delta}{\varepsilon}$. $\square$

Corollary 4.4. For each $\theta \in [0, 1]$ the inequality

$$\left| \frac{\varepsilon^2}{\varepsilon^2 + n_\varepsilon(\mathbf{k})} - \frac{1}{\tilde{m}(\mathbf{k})} \right| \lesssim \frac{\varepsilon^{1-\theta}}{(1 + |(k_1, \frac{k_2}{k_1})|^2)^{\frac{1}{2}(1+\theta)}}$$

holds uniformly over $|k_1|, |\frac{k_2}{k_1}| < \frac{\delta}{\varepsilon}$.

Proof. This result follows from Proposition 4.3 and the observation that $\varepsilon \lesssim \delta(1 + |(k_1, \frac{k_2}{k_2})|^2)^{-\frac{1}{2}}$ for $|k_1|, |\frac{k_2}{k_1}| < \frac{\delta}{\varepsilon}$. $\square$

Using Corollary 4.4, one can write equation (4.1) as

$$\zeta + F_\varepsilon(\zeta) = 0, \tag{4.2}$$

in which

$$F_\varepsilon(\zeta) = \tilde{m}(\mathbf{D})^{-1} \chi_\varepsilon(\mathbf{D}) \zeta^2 + \underbrace{T_{1,\varepsilon}(\zeta) + T_{2,\varepsilon}(\zeta)}_{= T_\varepsilon(\zeta)}$$

and

$$T_{1,\varepsilon}(\zeta) = \left(\varepsilon^2 (n_\varepsilon(\mathbf{D}) + \varepsilon^2)^{-1} - \tilde{m}(\mathbf{D})^{-1} \right) \chi_\varepsilon(\mathbf{D}) \zeta^2, \qquad T_{2,\varepsilon}(\zeta) = \varepsilon^2 (n_\varepsilon(\mathbf{D}) + \varepsilon^2)^{-1} S_\varepsilon(\zeta).$$

Proposition 4.5. Fix $\theta \in [0, 1]$. The mapping $T_\varepsilon \colon B_M(0) \subseteq Y_\varepsilon^1 \to Y_\varepsilon^{1+\theta}$ satisfies

$$|T_\varepsilon(\zeta)|_{Y^{1+\theta}} \lesssim \varepsilon^{1-\theta} |\zeta|_{Y^1}^2, \quad |dT_\varepsilon[\zeta]|_{\mathcal{L}(Y^1, Y^{1+\theta})} \lesssim \varepsilon^{1-\theta} |\zeta|_{Y^1}$$

for all $\zeta \in Y_\varepsilon^{1+\theta}$.

Proof. The result for $T_{1,\varepsilon}$ follows from the calculation

$$\left| \left(\varepsilon^2 (n_\varepsilon(\mathbf{D}) + \varepsilon^2)^{-1} - \tilde{m}(\mathbf{D})^{-1} \right) \chi_\varepsilon(\mathbf{D}) \zeta \rho \right|_{Y^{1+\theta}} \lesssim \varepsilon^{1-\theta} |\zeta \rho|_0 \lesssim \varepsilon^{1-\theta} |\zeta|_{L^4} |\rho|_{L^4} \lesssim \varepsilon^{1-\theta} |\zeta|_{Y^{1+\theta}} |\rho|_{Y^{1+\theta}}$$

for all $\zeta, \rho \in Y_\varepsilon^{1+\theta}$ (see Corollary 4.4 and Proposition 2.2(i)). Corollary 4.4 (with $\theta = 1$) also yields

$$\frac{\varepsilon^2}{n_\varepsilon(\mathbf{k}) + \varepsilon^2} \lesssim \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{-1},$$

and the result for $T_{2,\varepsilon}$ follows from this estimate and (3.7). $\square$

Remark 4.6. We can also consider T_ε as a mapping $T_\varepsilon \colon B_M(0) \subseteq Y_\varepsilon^{1+\theta} \to Y_\varepsilon^{1+\theta}$ with identical estimates since $\{Y_\varepsilon^r, |\cdot|_{Y^r}\}_{r \geq 0}$ is a scale of Banach spaces.

It is convenient to replace equation (4.2) with

$$\zeta + G_\varepsilon(\zeta) = 0,$$

where $G_\varepsilon(\zeta) = F_\varepsilon(\chi_\varepsilon(\mathbf{D})\zeta)$, and study it in the fixed space $Y^{1+\theta}$ for $\theta \in (\frac{1}{2}, 1)$ (the solution sets of the two equations evidently coincide); we choose $\theta > \frac{1}{2}$ so that $Y^{1+\theta}$ is embedded in $C_{\mathrm{b}}(\mathbb{R}^2)$ and $\theta < 1$ so that $T_\varepsilon(\zeta)$ vanishes in the limit $\varepsilon \to 0$. Note that the regularity index s for the space X must be taken larger than $r = 1 + \theta$ to preserve the embedding $X \hookrightarrow Y^r$ (see Lemma 2.1); in fact all desired properties are satisfed for $\frac{3}{2} < 1 + \theta < s < 2$. We establish Theorem 4.2 by applying Theorem 4.1 with

$$\mathcal{W} = Y_{\mathrm{e}}^{1+\theta} = \{\zeta \in Y^{1+\theta} : \zeta(x, y) = \zeta(-x, y) = \zeta(x, -y) \text{ for all } (x, y) \in \mathbb{R}^2\}, \tag{4.3}$$

$W_0 = B_M(0) \subseteq Y_{\mathrm{e}}^{1+\theta}$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0, and

$$\mathcal{G}(\zeta, \varepsilon) = \zeta + G_{|\varepsilon|}(\zeta), \tag{4.4}$$

where ε has been replaced by $|\varepsilon|$ to have $\mathcal{G}(\zeta, \varepsilon)$ defined for ε in a full neighbourhood of the origin in $\mathbb{R}$.

We begin by verifying that the functions $\zeta_k^\star$ belong to $Y_{\mathrm{e}}^{1+\theta}$.

Proposition 4.7. Each KP lump solution $\zeta_k^\star$ belongs to Y^2.

Proof. First note that $(\zeta_k^\star)^2$ belongs to $L^2(\mathbb{R}^2) = Y^0$ because $|\zeta_k^\star(x, y)| \lesssim (1 + x^2 + y^2)^{-1}$ for all $(x, y) \in \mathbb{R}^2$ (see Proposition 1.1(i)). Since $\zeta_k^\star$ satisfies

$$\zeta_k^\star + \tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star)^2 = 0$$

and $\tilde{m}(\mathbf{D})^{-1}$ is a lifting operator of order 2 for the scale $\{Y^r, |\cdot|_{Y^r}\}_{r \geq 0}$, one finds that $\zeta_k^\star \in Y^2$. $\qquad\square$

Observe that $\mathcal{G}(\cdot, \varepsilon)$ is a continuously differentiable function $B_M(0) \subseteq Y_{\mathrm{e}}^{1+\theta} \to Y_{\mathrm{e}}^{1+\theta}$ for each fixed $\varepsilon \geq 0$, so that

$$\lim_{\zeta \to \zeta_k^\star} |\mathrm{d}_1\mathcal{G}[\zeta, 0] - \mathrm{d}_1\mathcal{G}[\zeta_k^\star, 0]|_{\mathcal{L}(Y^{1+\theta}, Y^{1+\theta})} = 0.$$

The facts that

$$\lim_{\varepsilon \to 0} |\mathcal{G}(\zeta, \varepsilon) - \mathcal{G}(\zeta, 0)|_{Y^{1+\theta}} = 0, \quad \lim_{\varepsilon \to 0} |\mathrm{d}_1\mathcal{G}[\zeta, \varepsilon] - \mathrm{d}_1\mathcal{G}[\zeta, 0]|_{\mathcal{L}(Y^{1+\theta}, Y^{1+\theta})} = 0$$

uniformly over $\zeta \in B_M(0) \subseteq Y_{\mathrm{e}}^{1+\theta}$ are obtained from the equation

$$\mathcal{G}(\zeta, \varepsilon) - \mathcal{G}(\zeta, 0) = \tilde{m}(\mathbf{D})^{-1}\left(\chi_\varepsilon(\mathbf{D})(\chi_\varepsilon(\mathbf{D})\zeta)^2 - \zeta^2\right) + T_{|\varepsilon|}(\zeta)$$

using Proposition 4.5 and Corollary 4.10 below, which is a consequence of the next two lemmas.

Lemma 4.8. *Fix $\theta > \frac{1}{2}$. The estimate*

$$|\tilde{m}(\mathbf{D})^{-1}\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\rho)\big)|_{Y^{1+\theta}} \lesssim \varepsilon|\zeta|_{Y^{1+\theta}}|\rho|_{Y^{1+\theta}}$$

holds for all $\zeta, \rho \in Y^{1+\theta}$.

Proof. Recall that $\tilde{m}(\mathbf{D})^{-1}$ is a lifting operator of order 2 for the scale $\{Y^r, |\cdot|_{Y^r}\}_{r \geq 0}$ and that $\chi_\varepsilon(\mathbf{D})$ is a bounded projection on all subspaces of $L^2(\mathbb{R}^2)$. It follows that

$$\begin{aligned}
|\tilde{m}(\mathbf{D})^{-1}\chi_\varepsilon(\mathbf{D})&\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\rho)\big)|_{Y^{1+\theta}} \\
&\leq |\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\rho)\big)|_{L^2} \\
&\leq |((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\rho)|_{L^2} \\
&\leq |\zeta|_\infty|(\chi_\varepsilon(\mathbf{D}) - I)\rho|_{L^2} \\
&\leq |\zeta|_{Y^{1+\theta}}|(\chi_\varepsilon(\mathbf{D}) - I)\rho|_{L^2},
\end{aligned}$$

where the last line follows by the embedding $Y^{1+\theta} \hookrightarrow C_b^1(\mathbb{R}^2)$. To estimate $|\chi_\varepsilon(\mathbf{D}) - I)\zeta|_{L^2}$, note that

$$\mathbb{R}^2 \setminus C_\varepsilon \subset \underbrace{\left\{ (k_1, k_2) \colon |k_1| > \frac{\delta}{\varepsilon} \right\}}_{= C_\varepsilon^1} \cup \underbrace{\left\{ (k_1, k_2) \colon \left| \frac{k_2}{k_1} \right| > \frac{\delta}{\varepsilon} \right\}}_{= C_\varepsilon^2},$$

so that

$$
\begin{aligned}
|(\chi_\varepsilon(\mathbf{D}) - I)\zeta|_{L^2}^2 &= \int_{\mathbb{R}^2 \setminus C_\varepsilon} |\hat{\zeta}|^2 \, d\mathbf{k} \\
&\leq \int_{C_\varepsilon^1} |\hat{\zeta}|^2 \, d\mathbf{k} + \int_{C_\varepsilon^2} |\hat{\zeta}|^2 \, d\mathbf{k} \\
&\leq \frac{\varepsilon^2}{\delta^2} \int_{C_\varepsilon^1} k_1^2 |\hat{\zeta}|^2 \, d\mathbf{k} + \frac{\varepsilon^2}{\delta^2} \int_{C_\varepsilon^2} \frac{k_2^2}{k_1^2} |\hat{\zeta}|^2 \, d\mathbf{k} \\
&\leq \frac{2\varepsilon^2}{\delta^2} |\zeta|_{Y^1}^2. \qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Lemma 4.9. *Fix $\theta \in (0, 1)$. The estimate*

$$|\tilde{m}(\mathbf{D})^{-1}(\chi_\varepsilon(\mathbf{D}) - I)(\zeta\rho)|_{Y^{1+\theta}} \lesssim \varepsilon^{\frac{1-\theta}{2}} |\zeta|_{Y^1} |\rho|_{Y^1} \leq \varepsilon^{\frac{1-\theta}{2}} |\zeta|_{Y^{1+\theta}} |\rho|_{Y^{1+\theta}},$$

holds for all $\zeta, \rho \in Y^{1+\theta}$.

Proof. For $\nu \in \{k_1, \frac{k_2}{k_1}\}$ we find that

$$\left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{1+\theta} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{-2} \left(\frac{\varepsilon}{\delta} |\nu| \right)^{1-\theta} = \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} \left(\frac{|\nu|}{1 + k_1^2 + \frac{k_2^2}{k_1^2}} \right)^{1-\theta} \leq \frac{1}{2} \left(\frac{\varepsilon}{\delta} \right)^{1-\theta},$$

so that

$$
\begin{aligned}
|\tilde{m}(\mathbf{D})^{-1}&(\chi_\varepsilon(\mathbf{D}) - I)\zeta\rho|_{Y^{1+\theta}}^2 \\
&\lesssim \int_{C_\varepsilon^1 \cup C_\varepsilon^2} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{1+\theta} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{-2} |\mathcal{F}[\zeta\rho]|^2 \, d\mathbf{k} \\
&\lesssim \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} \int_{C_\varepsilon^1} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{1+\theta} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{-2} |k_1|^{1-\theta} |\mathcal{F}[\zeta\rho]|^2 \, d\mathbf{k} \\
&\quad + \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} \int_{C_\varepsilon^2} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{1+\theta} \left(1 + k_1^2 + \frac{k_2^2}{k_1^2} \right)^{-2} \left| \frac{k_2}{k_1} \right|^{1-\theta} |\mathcal{F}[\zeta\rho]|^2 \, d\mathbf{k} \\
&\leq \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} |\zeta\rho|_{L^2}^2 \\
&\lesssim \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} |\zeta|_{L^4}^2 |\rho|_{L^4}^2 \\
&\lesssim \left(\frac{\varepsilon}{\delta} \right)^{1-\theta} |\zeta|_{Y^1}^2 |\rho|_{Y^1}^2,
\end{aligned}
$$

where we have used Parseval's theorem, the Cauchy-Schwarz inequality and the embedding $Y^1 \hookrightarrow L^4(\mathbb{R}^2)$ (see Proposition 2.2). $\qquad\square$

Corollary 4.10. Fix $\theta \in (\frac{1}{2}, 1)$. The estimate

$$\left| \tilde{m}(\mathbf{D})^{-1} \Big(\chi_\varepsilon(\mathbf{D}) \big((\chi_\varepsilon(\mathbf{D})\zeta)(\chi_\varepsilon(\mathbf{D})\rho) \big) - \zeta\rho \Big) \right|_{Y^{1+\theta}} \lesssim \varepsilon^{\frac{1-\theta}{2}} |\zeta|_{Y^{1+\theta}} |\rho|_{Y^{1+\theta}}$$

holds for all $\zeta, \rho \in Y^{1+\theta}$.

Proof. This result is obtained by writing

$$\begin{aligned}
\tilde{m}(\mathbf{D})^{-1} &\Big(\chi_\varepsilon(\mathbf{D}) \big((\chi_\varepsilon(\mathbf{D})\zeta)(\chi_\varepsilon(\mathbf{D})\rho) \big) - \zeta\rho \Big) \\
&= \tfrac{1}{2} \tilde{m}(\mathbf{D})^{-1} \chi_\varepsilon(\mathbf{D}) \big(((\chi_\varepsilon(\mathbf{D})+1)\zeta)((\chi_\varepsilon(\mathbf{D})-1)\rho) \big) \\
&\quad + \tfrac{1}{2} \tilde{m}(\mathbf{D})^{-1} \chi_\varepsilon(\mathbf{D}) \big(((\chi_\varepsilon(\mathbf{D})+1)\rho)((\chi_\varepsilon(\mathbf{D})-1)\zeta) \big) \\
&\quad + \tilde{m}(\mathbf{D})^{-1} (\chi_\varepsilon(\mathbf{D})-1)(\zeta\rho),
\end{aligned}$$

and applying Lemma 4.9 to the first two terms on the right-hand side and Lemma 4.10 to the third. $\square$

It thus remains to show that

$$\mathrm{d}_1 \mathcal{G}[\zeta_k^\star, 0] = I + 2\tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star \cdot)$$

is an isomorphism; this fact is a consequence of the following result.

Lemma 4.11. *The operator* $\tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star \cdot) \colon Y^{1+\theta} \to Y^{1+\theta}$ *is compact.*

Proof. Let $\{\zeta_j\}$ be a sequence which is bounded in Y^1. We can find a subsequence of $\{\zeta_j\}$ (still denoted by $\{\zeta_j\}$) which converges weakly in $L^2(\mathbb{R}^2)$ (because $\{\zeta_j\}$ is bounded in $L^2(\mathbb{R}^2)$) and strongly in $L^2(|(x,y)| < n)$ for each $n \in \mathbb{N}$ (by Proposition 2.2(ii) and a 'diagonal' argument). Denote the limit by ζ_∞. Since

$$|\zeta_k^\star \zeta_j - \zeta_k^\star \zeta_\infty|_{L^2(|(x,y)|<n)} \leq |\zeta_k^\star|_\infty |\zeta_j - \zeta_\infty|_{L^2(|(x,y)|<n)} \to 0$$

as $j \to \infty$ for each $n \in \mathbb{N}$ and

$$\sup_j |\zeta_k^\star \zeta_j|_{L^2(|(x,y)|>n)} \leq \sup_{|(x,y)|>n} |\zeta_k^\star(x,y)| \sup_j |\zeta_j|_{L^2} \to 0$$

as $n \to \infty$ we conclude that $\{\zeta_k^\star \zeta_j\}$ converges to $\zeta_k^\star \zeta_\infty$ as $j \to \infty$ in $L^2(\mathbb{R}^2)$. It follows that $\zeta \mapsto \zeta_k^\star \zeta$ is compact $Y^1 \to L^2(\mathbb{R})$ and hence $Y^{1+\theta} \to L^2(\mathbb{R})$; the result follows from this fact and the observation that $\tilde{m}(\mathbf{D})^{-1}$ is continuous $L^2(\mathbb{R}^2) \to Y^2 \hookrightarrow Y^{1+\theta}$. $\square$

Corollary 4.12. *The operator* $I + 2\tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star \cdot)$ *is an isomorphism* $Y_{\mathrm{e}}^{1+\theta} \to Y_{\mathrm{e}}^{1+\theta}$.

Proof. The previous result shows that $I + 2\tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star \cdot) \colon Y_{\mathrm{e}}^{1+\theta} \to Y_{\mathrm{e}}^{1+\theta}$ is Fredholm with index 0; it therefore remains to show that it is injective. Suppose that $\zeta \in Y_{\mathrm{e}}^{1+\theta}$ satisfies

$$\zeta + 2\tilde{m}(\mathbf{D})^{-1}(\zeta_k^\star \zeta) = 0. \tag{4.5}$$

It follows that

$$k_1 \hat{\zeta} = \frac{-2k_1^3}{k_1^2 + \frac{1}{2}(\beta - \frac{1}{3})k_1^4 + k_2^2} \mathcal{F}[\zeta_k^\star \zeta], \qquad k_2 \hat{\zeta} = \frac{-2k_1^2 k_2}{k_1^2 + \frac{1}{2}(\beta - \frac{1}{3})k_1^4 + k_2^2} \mathcal{F}[\zeta_k^\star \zeta]$$

and hence $\zeta \in H^{j+1}(\mathbb{R}^2)$ whenever $\zeta_k^\star \zeta \in H^j(\mathbb{R}^2)$. Since $\zeta \in L^2(\mathbb{R}^2)$ and $\zeta \in H^j(\mathbb{R}^2)$ implies $\zeta_k^\star \zeta \in H^j(\mathbb{R}^2)$ because $\zeta_k^\star \in C_{\mathrm{b}}^j(\mathbb{R}^2)$, the space of smooth functions on $\mathbb{R}^2$ with bounded derivatives up to order j, we find by bootstrapping that $\zeta \in H^\infty(\mathbb{R}^2)$.

Since ζ is smooth and satisfies (4.5), it satisfies the linear equation

$$\left((\beta - \tfrac{1}{3})\zeta_{xx} + 2\zeta + 2(\zeta_k^\star \zeta)\right)_{xx} - \zeta_{zz} = 0,$$

and this equation has only the trivial smooth, decaying, symmetric solution (see Lemma 1.1(iii)).   □

Having completed the proof of Theorem 4.2, we now finalise the proof of Theorem 1.2 by tracing back the steps in the reduction procedure to construct solutions to (1.3) which are uniformly approximated by a suitable scaling of $\zeta_k^\star$.

Lemma 4.13. *The formula*

$$u = u_1 + u_2(u_1), \quad u_1(x, y) = \varepsilon^2 \zeta_k^\varepsilon(\varepsilon x, \varepsilon^2 y)$$

defines a smooth solution to the steady FDKP equation (1.3) *which satisfies the estimate*

$$u(x, y) = \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + o(\varepsilon^2)$$

uniformly over $(x, y) \in \mathbb{R}^2$.

Proof. Theorem 4.2 implies that

$$|\zeta_k^\varepsilon - \zeta_k^\star|_\infty = o(1)$$

as $\varepsilon \to 0$ because of the embedding $Y^{1+\theta} \hookrightarrow C_b(\mathbb{R}^2)$ (see Proposition 2.3). It follows that

$$
\begin{aligned}
u_1(x, y) &= \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + \varepsilon^2(\zeta_k^\varepsilon - \zeta_k^\star)(\varepsilon x, \varepsilon^2 y) \\
&= \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + o(\varepsilon^2)
\end{aligned}
$$

as $\varepsilon \to 0$ uniformly over $(x, y) \in \mathbb{R}^2$, while

$$|u_2(u_1)|_\infty \lesssim |u_2(u_1)|_{X_2} \lesssim \varepsilon |u_1|_\varepsilon^2 \lesssim \varepsilon^3$$

because $|u_2(u_1)|_{X_2} \lesssim \varepsilon |u_1|_\varepsilon^2$ and $|u_1|_\varepsilon = \varepsilon |\zeta|_{Y^1}$ with $\zeta \in B_M(0) \subseteq Y_\varepsilon^1$. The fact that $u = u_1 + u_2(u_1)$ is smooth follows from Proposition 3.4.   □

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