An existence theory for solitary waves on a ferrofluid jet

M. D. Groves* D. Nilsson† L. Schütz*

Abstract

We discuss axisymmetric solitary waves on the surface of an otherwise cylindrical ferrofluid jet surrounding a stationary metal rod. The ferrofluid, which is governed by a general (nonlinear) magnetisation law, is subject to an azimuthal magnetic field generated by an electric current flowing along the rod. We treat the governing equations using a modification of the Zakharov-Craig-Sulem formulation for water waves, reducing the problem to a single nonlocal equation for the free-surface elevation variable η. The nonlocality in the equation takes the form of a Dirichlet-Neumann operator whose analyticity (in standard function spaces) is demonstrated by studying its defining boundary-value problem in newly introduced Sobolev spaces for radial functions. Using rudimentary fixed-point arguments and Fourier analysis we rigorously reduce the equation for η to a perturbation of a Korteweg-de Vries equation (for strong surface tension) or a nonlinear Schrödinger equation (for weak surface tension), both of which have nondegenerate explicit solitary-wave solutions. The existence theory is completed using an appropriate version of the implicit-function theorem.

1 Introduction

1.1 The hydrodynamic problem

We consider the inviscid, incompressible and irrotational flow of a ferrofluid of unit density in the region

$$S_1 = \{0 < r < R + \eta(\theta, z, t)\}$$

bounded by the free surface $\{r = R + \eta(\theta, z, t)\}$ and a current-carrying wire at $\{r = 0\}$ (see Figure 1). Here (r, θ, z) are the usual cylindrical polar coordinates, t is time, R is a positive constant which represents the radius of the jet without any current flow, and η is a function of (θ, z, t). The magnetic field generated by the wire is static and the region

$$S_2 = \{r > R + \eta(\theta, z, t)\}$$

is assumed to be a vacuum. The irrotational magnetic and solenoidal induction fields in S_1 and S_2 are denoted by respectively $\boldsymbol{H}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{H}_2$, $\boldsymbol{B}_2$, while the irrotational, solenoidal velocity field of the fluid in S_1 is denoted by $\boldsymbol{v}$. The interdependence between the fields is given by the formulae

$$\boldsymbol{B}_1 = \mu_0(\boldsymbol{H}_1 + \boldsymbol{M}_1(\boldsymbol{H}_1)), \qquad \boldsymbol{B}_2 = \mu_0 \boldsymbol{H}_2,$$

where μ_0 is the magnetic permeability of free space,

$$\boldsymbol{M}_1(\boldsymbol{H}_1) = m_1(|\boldsymbol{H}_1|)\frac{\boldsymbol{H}_1}{|\boldsymbol{H}_1|}$$

is the given magnetic intensity of the ferrofluid and $m_1(\boldsymbol{H}_1)$ is a nonnegative function.

The ferrohydrodynamic problem was formulated in terms of magnetic potential functions ψ_1, ψ_2 and a velocity potential ϕ such that

$$\boldsymbol{H}_1 = -\nabla\psi_1, \quad \boldsymbol{H}_2 = -\nabla\psi_2, \quad \boldsymbol{v} = \nabla\phi$$

by Groves & Nilsson [12, §2] following the theory given by Rosensweig [17, §§5.1–5.2]. The governing equations are

$$\nabla \cdot (\mu(|\nabla\psi_1|\nabla\psi_1)) = 0, \qquad 0 < r < R + \eta(\theta, z, t), \tag{1}$$

$$\Delta\psi_2 = 0, \qquad r > R + \eta(\theta, z, t), \tag{2}$$

$$\Delta\phi = 0, \qquad 0 < r < R + \eta(\theta, z, t), \tag{3}$$

*FR Mathematik, Universität des Saarlandes, Postfach 151150, 66041 Saarbrücken, Germany
†Department of Mathematics, Linnaeus University, Växjö, Sweden

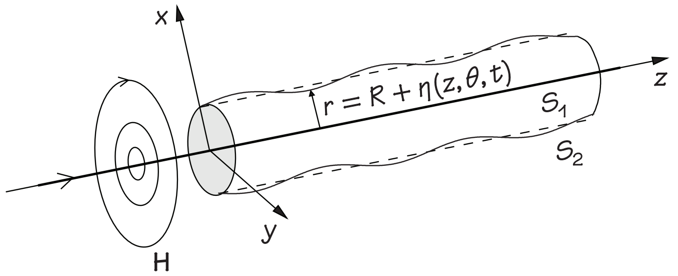

Figure 1: Waves on the surface of a ferrofluid jet surrounding a current-carrying wire

where

$$\mu(s) = 1 + \frac{m(s)}{s},$$

with boundary conditions

$$\psi_2 - \psi_1 = 0, \tag{4}$$

$$\psi_{2n} - \mu(|\nabla\psi_1|)\psi_{1n} = 0, \tag{5}$$

$$-\eta_t + \phi_r - \frac{1}{r^2}\phi_\theta\eta_\theta - \phi_z\eta_z = 0, \tag{6}$$

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 - \mu_0\nu(|\nabla\psi_1|) + 2\sigma\kappa - \tfrac{1}{2}\mu_0(\mu(|\nabla\psi_1|) - 1)^2 = c_0 \tag{7}$$

at $r = R + \eta(\theta, z, t)$, where 2κ is the mean curvature of the surface, σ is the coefficient of surface tension and c_0 is a constant arising from integration of the (magnetic) Euler equation. Equations (1)–(3) state that $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and $\boldsymbol{v}$ are solenoidal, equations (4), (5) state that the magnetic and induction fields are continuous at the surface, while equation (6) is the the hydrokinematic boundary condition that fluid particles on the surface remain there and equation (7) is the hydrodynamic boundary condition which balances the forces at the surface.

The constant c_0 is selected so that

$$\mathbf{H}_1 = \frac{J}{2\pi r}\mathbf{e}_\theta, \qquad \mathbf{H}_2 = \frac{J}{2\pi r}\mathbf{e}_\theta, \qquad \mathbf{v} = \mathbf{0}, \qquad \eta = 0$$

(that is $\psi_1 = \psi_2 = -J\theta/2\pi$, $\phi = 0$, $\eta = 0$) is a solution to the above equations (corresponding to a uniform magnetic field and a circular cylindrical jet with radius R); we therefore set $c_0 = -\mu_0\nu(J/2\pi r) + \sigma/R$. Seeking axisymmetric waves for which η and ϕ are independent of θ, one finds that $\psi_1 = \psi_2 = -J\theta/2\pi$, so that the hydrodynamic problem decouples from the magnetic problem and is given by

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad 0 < r < R + \eta(z, t)$$

and

$$\begin{aligned}
&-\eta_t + \phi_r - \phi_z\eta_z = 0,\\
&\phi_t + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \mu_0\nu\left(\frac{J}{2\pi(R+\eta)}\right)\\
&\qquad + \mu_0\nu\left(\frac{J}{2\pi R}\right) + \frac{\sigma}{(R+\eta)(1+\eta_z^2)^{1/2}} - \frac{\sigma\eta_{zz}}{(1+\eta_z^2)^{3/2}} - \frac{\sigma}{R} = 0
\end{aligned}$$

at $r = R + \eta(z, t)$, where we have used the formula

$$2\kappa = \frac{-(R+\eta)^2(1+\eta_z^2) + (R+\eta)^3\eta_{zz}}{(R+\eta)^{3/2}(1+\eta_z^2)^{3/2}}.$$

This initial-value problem has been studied by Wang & Yang [18], but here we concentrate upon travelling waves. Introducing dimensionless variables

$$(\hat{z}, \hat{r}) := \frac{1}{R}(z, r), \qquad \hat{t} = \frac{\sigma^{1/2}}{R^{3/2}}t, \qquad \hat{\phi} := \frac{1}{(\sigma R)^{1/2}}\phi, \qquad \hat{\eta} := \frac{1}{R}\eta$$

and functions

$$\hat{m}_1(s) := \frac{2\pi R}{J\chi} m_1\left(\frac{J}{2\pi R}s\right), \qquad \hat{\nu}(s) := \frac{4\pi^2 R^2}{J^2\chi}\nu\left(\frac{J}{2\pi R}s\right),$$

where $\chi = (2\pi R/J)m_1(J/2\pi R)$ and $\hat{m}(1) = \hat{\nu}'(1) = 1$, and looking for travelling-wave solutions of the form

$$\phi(r,z,t) = \phi(r, z-ct), \qquad \eta(z,t) = \eta(z-ct),$$

we arrive at the equations

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad 0 < r < 1 + \eta(z,t), \tag{8}$$

and

$$c\eta_z + \phi_r - \phi_z\eta_z = 0, \tag{9}$$

$$-c\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \gamma\left(\nu\left(\frac{1}{1+\eta}\right) - \nu(1)\right) + \left(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\right) = 0 \tag{10}$$

at $r = 1 + \eta(z,t)$, where

$$\gamma = \frac{\mu_0 J^2 \chi}{4\pi^2\sigma R^2}.$$

Solitary waves are nontrivial solutions to (8)–(10) which are evanescent as $|z| \to \infty$.

1.2 The main results

We treat equations (8)–(10) using a modification of the Zakharov-Craig-Sulem formulation for water waves (Zakharov [19], Craig & Sulem [7]), thus reducing the problem to a single non-local equation for η by introducing a *Dirichlet-Neumann operator* informally defined as follows (see Xu & Wang [20] for a similar approach for the time-dependent problem and Blyth & Parau [4] for an alternative non-local reformulation). Fix $\Phi = \Phi(z)$, let ϕ be the unique solution of the Dirichlet boundary-value problem

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad 0 < r < 1 + \eta, \tag{11}$$

$$\phi = \Phi \qquad r = 1 + \eta, \tag{12}$$

and define

$$\begin{aligned} G(\eta)\Phi &:= (1+\eta)(1+\eta_z^2)^{1/2}\frac{\partial\phi}{\partial n}\Big|_{r=1+\eta} \\ &= (1+\eta)(\phi_r - \eta_z\phi_z)\big|_{r=1+\eta}. \end{aligned}$$

Equations (9) and (10) can be rewritten as

$$c\eta_z + \frac{G(\eta)\Phi}{1+\eta} = 0 \tag{13}$$

and

$$\begin{aligned} &-c\Phi_z + \tfrac{1}{2}\Phi_z^2 - \frac{1}{2(1+\eta_z^2)}\left(\eta_z\Phi_z + \frac{G(\eta)\Phi}{1+\eta}\right)^2 \\ &\qquad - \gamma\left(\nu\left(\frac{1}{1+\eta}\right) - \nu(1)\right) + \left(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\right) = 0, \end{aligned} \tag{14}$$

and by substituting $\Phi = -cG(\eta)^{-1}(\eta_z + \eta\eta_z)$ from (13) into (14), we arrive at

$$\mathcal{K}(\eta) - c^2\mathcal{L}(\eta) = 0, \tag{15}$$

where

$$\mathcal{K}(\eta) = -\gamma\left(\nu\left(\frac{1}{1+\eta}\right) - \nu(1)\right) + \left(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\right), \tag{16}$$

$$\begin{aligned} \mathcal{L}(\eta) = &-\tfrac{1}{2}(K(\eta)\eta + \tfrac{1}{2}K(\eta)\eta^2)^2 + \frac{1}{2(1+\eta_z^2)}(\eta_z - \eta_z K(\eta)\eta - \tfrac{1}{2}\eta_z K(\eta)\eta^2)^2 \\ &+ K(\eta)\eta + \tfrac{1}{2}K(\eta)\eta^2 \end{aligned} \tag{17}$$

and

$$K(\eta)\xi = -(G(\eta)^{-1}\xi_z)_z. \tag{18}$$

Equation (15) is equivalent to (8)–(10); the velocity potential is recovered by setting $\Phi = -cG(\eta)^{-1}(\eta_z + \eta\eta_z)$ and solving (11), (12).

Our task is therefore to find nontrivial solutions to (15) which satisfy $\eta(z) \to 0$ as $z \to \pm\infty$, and we prove the following results.

Theorem 1.1 *Suppose that* $1 < \gamma < 9$ *and* $c^2 = c_0^2(1-\varepsilon^2)$. *For each sufficiently small value of* $\varepsilon > 0$ *there exists a symmetric Korteweg-de Vries solitary-wave solution of* (15) *which satisfies*

$$\eta(z) = \varepsilon^2 \zeta_{\mathrm{KdV}}(\varepsilon z) + o(\varepsilon^2)$$

uniformly over $z \in \mathbb{R}$, *where*

$$\zeta_{\mathrm{KdV}}(Z) = -\frac{3}{2d_0}\operatorname{sech}^2\left(2\left(\frac{c_0^2}{9-\gamma}\right)^{1/2} Z\right) \tag{19}$$

and

$$d_0 = \frac{1}{2c_0^2}\left(\frac{3}{2}\gamma - \frac{1}{2}\gamma\nu''(1) - \frac{3}{2}\right), \qquad c_0^2 = \frac{1}{2}(\gamma - 1).$$

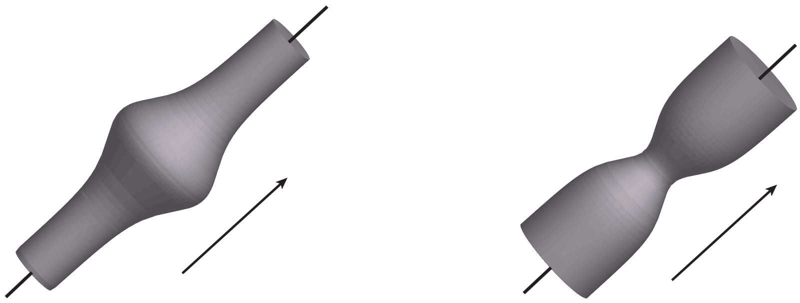

Figure 2: Korteweg-de Vries solitary waves of elevation (left) and of depression (right) depending on the sign of d_0

Theorem 1.2 *Suppose that* $\omega > 0$,

$$\gamma = 1 - \omega^2 + \frac{2\omega f(\omega)}{f'(\omega)}, \qquad c_0^2 = \frac{2\omega}{f'(\omega)}, \qquad f(\omega) = \frac{\omega I_0(\omega)}{I_1(\omega)},$$

where I_ν *is the modified Bessel function of the first kind and order* ν, *and* $c^2 = c_0^2(1-\varepsilon^2)$. *For each sufficiently small value of* $\varepsilon > 0$ *there exist two symmetric nonlinear Schrödinger solitary-wave solutions of* (15) *which satisfy*

$$\eta(z) = \pm\varepsilon\zeta_{\mathrm{NLS}}(\varepsilon z)\cos(\omega z) + o(\varepsilon) \tag{20}$$

uniformly over $z \in \mathbb{R}$, *where*

$$\zeta_{\mathrm{NLS}}(Z) = \left(\frac{2a_2}{a_3}\right)^{1/2} \operatorname{sech}\left(\left(\frac{a_2}{a_1}\right)^{1/2} Z\right) \tag{21}$$

and a_1, a_2, a_3 *are positive constants which depend upon* ω.

Axisymmetric solitary waves have also been investigated using model equations by Bashtovoi, Rex & Foiguel [2] and Rannacher & Engel [16], experimentally by Bourdin, Bacri & Falcon [5] and numerically by Blyth & Parau [3], Guyenne & Parau [13], Doak & Vanden-Broeck [9] and Xu & Wang [20]. Furthermore, using spatial dynamical-systems methods Groves & Nilsson [12] have given a rigorous existence theory for multiple types of solitary waves (including those in Theorems 1.1 and 1.2).

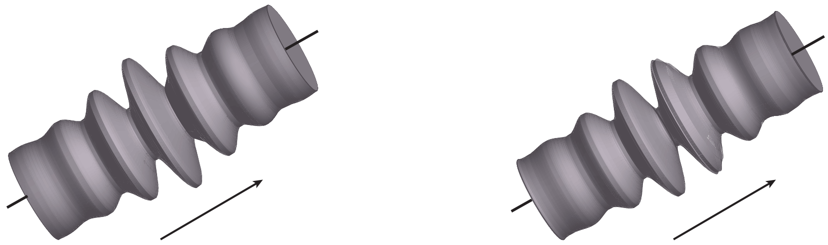

Figure 3: Nonlinear Schrödinger solitary waves of elevation (left) and of depression (right) depending on the sign in equation (20)

1.3 Weakly nonlinear theory

It is instructive to present a heuristic argument as a motivation for Theorems 1.1 and 1.2, beginning with the linearised problem. Linearising equation (15) yields

$$(\gamma-1)\eta-\eta_{zz}-c^2K_0\eta=0, \tag{22}$$

where $K_0=f(D)$ and

$$f(|k|)=\frac{|k|I_0(|k|)}{I_1(|k|)}.$$

here we used the notation

$$h(D)\xi=\mathcal{F}[h(k)\hat{\xi}], \qquad \hat{\xi}=\mathcal{F}[\xi],$$

for the Fourier multiplier defined by h, where $\mathcal{F}$ is the one-dimensional Fourier transform defined by

$$\mathcal{F}[\xi](k)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\xi(z)\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z$$

and $D=-\mathrm{i}\partial_z$. Seeking solutions of (22) of the form $\eta(z)=\cos(kz)$ ('sinusoidal wave trains'), we obtain the dispersion relation

$$c^2=\frac{\gamma-1+k^2}{f(k)},$$

which describes the relation between the wave number $k\geq 0$ and the wave speed $c\geq 0$. In Appendix A we show that c^2 is a strictly monotone increasing function of k for $1<\gamma\leq 9$, while for $\gamma>9$ it has a unique local maximum at $k=0$ and a unique global minimum at $k=\omega>0$ (the formula $\gamma=1-\omega^2+2\omega f(\omega)/f'(\omega)$ defines a bijection between the values of $\gamma\in(9,\infty)$ and $\omega\in(0,\infty)$). In both cases we denote its global minimum by c_0^2, so that

$$c_0^2=\begin{cases} c^2(0)=\frac{1}{2}(\gamma-1), & 1<\gamma\leq 9,\\ c^2(\omega)=\dfrac{2\omega}{f(\omega)}, & \gamma>9\end{cases}$$

(see Figure 4).

Using c as a bifurcation parameter, we expect branches of small-amplitude solitary waves to bifurcate at $c=c_0$ (where the linear group and phase speeds are equal) into the region $\{c<c_0\}$ where linear periodic wave trains are not supported (see Dias & Kharif [8, §3]). In the case $1<\gamma<9$, one writes $c^2=c_0^2(1-\varepsilon^2)$, where ε is a small positive number, substitutes the Ansatz

$$\eta(z)=\varepsilon^2\zeta_1(Z)+\varepsilon^4\zeta_2(Z)+\cdots, \tag{23}$$

where $Z=\varepsilon z$, into equation (15), and finds that ζ_1 satisfies the stationary Korteweg-de Vries equation

$$(\tfrac{1}{8}\gamma-\tfrac{9}{8})\zeta_{ZZ}+2c_0^2\zeta+2c_0^2d_0\zeta^2=0, \tag{24}$$

which has the explicit (symmetric) solitary-wave solution ζ_{KdV} given in Theorem 1.1. In the case $\gamma>9$, one writes $c^2=c_0^2(1-\varepsilon^2)$, uses the Ansatz

$$\eta(z)=\tfrac{1}{2}\varepsilon\big(\zeta_1(Z)\mathrm{e}^{\mathrm{i}\omega x}+\overline{\zeta_1(Z)}\mathrm{e}^{-\mathrm{i}\omega x}\big)+\varepsilon^2\zeta_0(Z)+\tfrac{1}{2}\varepsilon^2\big(\zeta_2(Z)\mathrm{e}^{2\mathrm{i}\omega z}+\overline{\zeta_2(Z)}\mathrm{e}^{-2\mathrm{i}\omega z}\big)+\cdots, \tag{25}$$

where $Z = \varepsilon z$ and $\gamma = 1 - \omega^2 + 2\omega f(\omega)/f'(\omega)$, and finds that ζ_1 satisfies the stationary nonlinear Schrödinger equation

$$-a_1\zeta_{ZZ} + a_2\zeta - a_3|\zeta|^2\zeta = 0, \tag{26}$$

which has the (symmetric) solitary-wave solutions $\pm\zeta_{\mathrm{NLS}}$ given in Theorem 1.2. Details of these calculations are given in Appendix B.

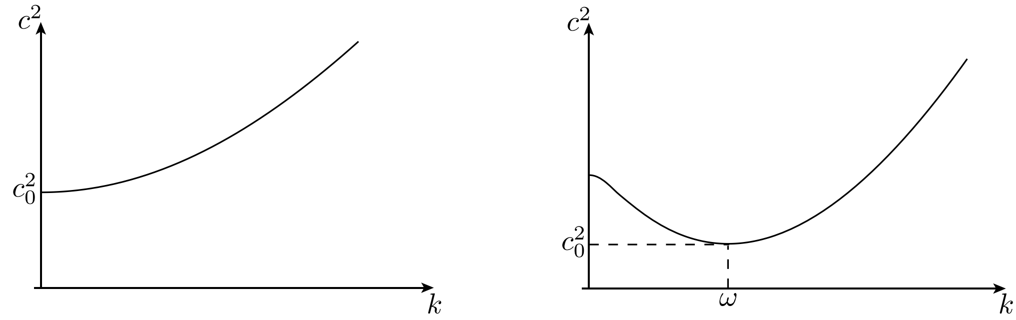

Figure 4: Dispersion relation in the cases $1 < \gamma \leq 9$ (left) and $\gamma > 9$ (right); the minimum value of c^2 is denoted by c_0^2

1.4 Methodology

In this paper we rigorously confirm the results of the weakly nonlinear theory described above. The Ansätze (23) and (25) suggest that the Fourier transform of a solitary wave is concentrated near the points $k = \pm\omega$ (which coincide at $k = 0$ when $1 < \gamma < 9$). Indeed, writing $c^2 = c_0^2(1 - \varepsilon^2)$, one finds that the linearisation of (15) at $\varepsilon = 0$ is

$$g(D)\eta = 0,$$

where

$$g(k) := \gamma - 1 + k^2 - c_0^2 f(k) \geq 0, \qquad k \in \mathbb{R},$$

with equality precisely when $k = \pm\omega$ (so that $g(\omega) = g'(\omega) = 0$ and $g''(\omega) > 0$). We therefore decompose η into the sum of functions η_1 and η_2 whose Fourier transforms $\hat{\eta}_1$ and $\hat{\eta}_2$ are supported in the region $S = (-\omega - \delta, -\omega + \delta) \cup (\omega - \delta, \omega + \delta)$ (with $\delta \in (0, \frac{\omega}{3})$) and its complement (see Figure 5), so that $\eta_1 = \chi(D)\eta$, $\eta_2 = (1 - \chi(D))\eta$, where χ is the characteristic function of the set S (note that $S = (-\delta, \delta)$ if $\omega = 0$). Decomposing (15) into

$$\begin{aligned} \chi(D)\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) &= 0, \\ (1 - \chi(D))\left(\mathcal{K}(\eta_1 + \eta_2) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2)\right) &= 0, \end{aligned}$$

one finds that the second equation can be solved for η_2 as a function of η_1 for sufficiently small values of $\varepsilon > 0$; substituting $\eta_2 = \eta_2(\eta_1)$ into the first yields the reduced equation

$$\chi(D)\left(\mathcal{K}(\eta_1 + \eta_2(\eta_1)) - c_0^2(1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2(\eta_1))\right) = 0$$

for η_1 (see Section 3).

Figure 5: (a) The support of $\hat{\eta}_1$ is contained in the set S, where $S = (-\delta, \delta)$ for $1 < \gamma < 9$ (left) and $S = (-\omega - \delta, -\omega + \delta) \cup (\omega - \delta, \omega + \delta)$ for $\gamma > 9$ (right).

Finally, the scaling

$$\eta_1(z) = \varepsilon^2\zeta(Z), \qquad Z = \varepsilon z, \tag{27}$$

transforms the reduced equation into

$$\varepsilon^{-2}g(\varepsilon D)\zeta + 2c_0^2\zeta + 2c_0^2 d_0\chi_0(\varepsilon D)\zeta^2 + O(\varepsilon^{1/2}) = 0 \tag{28}$$

for $1 < \gamma < 9$, while the scaling

$$\eta_1(z) = \tfrac{1}{2}\varepsilon\zeta(Z)\mathrm{e}^{\mathrm{i}\omega z} + \tfrac{1}{2}\varepsilon\overline{\zeta(Z)}\mathrm{e}^{-\mathrm{i}\omega z}, \qquad Z = \varepsilon z, \tag{29}$$

transforms the reduced equation into

$$\varepsilon^{-2}g(\omega + \varepsilon D)\zeta + a_2\zeta - a_3\chi_0(\varepsilon D)(|\zeta|^2\zeta) + O(\varepsilon^{1/2}) = 0 \tag{30}$$

for $\gamma > 9$; here χ_0 is the characteristic function of the set $(-\delta, \delta)$, the symbol D now means $-\mathrm{i}\partial_Z$ and precise estimates for the remainder terms are given in Section 3. Equations (28) and (30) are *full dispersion* versions of (perturbed) stationary Korteweg-de Vries and nonlinear Schrödinger equations since they retain the linear part of the original equation (15); the fully reduced model equations (24) and (26) are recovered from them in the formal limit $\varepsilon \to 0$.

The functions ζ_{KdV} and $\pm\zeta_{\mathrm{NLS}}$ are nondegenerate solutions of (24) and (26) in the sense that the only bounded solutions of their linearisations at ζ_{KdV} and $\pm\zeta_{\mathrm{NLS}}$ are respectively $\zeta_{\mathrm{KdV},Z}$ and $\pm\zeta_{\mathrm{NLS},Z}$, $\pm\mathrm{i}\zeta_{\mathrm{NLS}}$. Equation (15) is invariant under the reflection $\eta(z) \mapsto \eta(-z)$, and the reduction procedure preserves this property: the reduced equation for η_1 is invariant under the reflection $\eta_1(z) \mapsto \eta_1(-z)$, so that (24) and (26) are invariant under respectively $\zeta(Z) \mapsto \zeta(-Z)$ and $\zeta(Z) \mapsto \overline{\zeta(-Z)}$. Restricting to spaces of symmetric functions thus eliminates the antisymmetric solutions $\zeta_{\mathrm{KdV},Z}$ and $\pm\zeta_{\mathrm{NLS},Z}$, $\pm\mathrm{i}\zeta_{\mathrm{NLS}}$ of the linearised equations, and in Section 4 solutions to (28) and (30) are constructed as perturbations of ζ_{KdV} and $\pm\zeta_{\mathrm{NLS}}$ by formulating them as fixed-point equations and using an appropriate version of the implicit-function theorem.

This method has been used for the classical water-wave problem by Groves [10], and since many of the details in the derivation and solution of our reduced equations are similar to those in that reference we keep Sections 3 and 4 concise. We begin our analysis by showing that the functionals $\mathcal{K}$ and $\mathcal{L}$ in equation (15) depend analytically upon η in a suitable sense (see Buffoni & Toland [6] for a treatise on analytic functions in Banach spaces), which of course entails rigorously defining the operator K given by (18) and demonstrating its analyticity. This step, the details of which are given in Section 2, differs significantly from the corresponding step in reference [10]; in particular it is necessary to study an axisymmetric boundary-value problem using novel function spaces and carefully estimate a Green's function defined in terms of modified Bessel functions.

1.5 Function spaces

In addition to the familiar Sobolev spaces

$$H^s(\mathbb{R}) = \left\{ \eta \in \mathscr{S}'(\mathbb{R}) \,\Big|\, \|\eta\|_s^2 := \int_{\mathbb{R}} (1+k^2)^s |\hat{\eta}(k)|^2 \,\mathrm{d}k < \infty \right\}, \qquad s \geq 0$$

we use the variants

$$H_\varepsilon^s(\mathbb{R}) = \chi_0(\varepsilon D) H^s(\mathbb{R}), \qquad s \geq 0$$

and

$$\mathcal{Z} = \left\{ \eta \in \mathscr{S}'(\mathbb{R}) \,\big|\, \|\eta\|_{\mathcal{Z}} := \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 < \infty \right\},$$

where

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1 - \chi(D))\eta$$

(see Section 1.4 above). Note in particular the estimate

$$\|\eta_1\|_{j,\infty} \leq \|k^j \hat{\eta}(k)\|_{L^1(\mathbb{R})} \lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R})},$$

which holds because $\hat{\eta}_1$ has compact support, and implies in particular that

$$\|\eta\|_{1,\infty} \leq \|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty} \lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 = \|\eta\|_{\mathcal{Z}}. \tag{31}$$

Our analyticity result for the operator K defined by equation (18) is given in terms of the space $\mathcal{Z}$.

Lemma 1.3 *The mapping $K : \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ is analytic at the origin.*

This lemma is proved in Section 2, where we work with the equivalent definition

$$K(\eta)\xi = -(\tilde{\phi}|_{r=1+\eta})_z, \tag{32}$$

where $\tilde{\phi}$ is the axisymmetric solution of the Neumann boundary-value problem

$$\Delta\tilde{\phi} = 0, \qquad 0 < r < 1+\eta,$$
$$(1+\eta)(1+\eta_z^2)^{1/2}\frac{\partial\tilde{\phi}}{\partial n} = \xi_z, \qquad r = 1+\eta$$

(which is unique up to additive constants). To solve this boundary-value problem it is obviously necessary to study axisymmetric functions in the ferrofluid domain $\{0 < r < 1+\eta\}$. For this purpose we use the radial function spaces introduced by Groves & Hill [11] for functions defined on the reference domain $\{0 < r < 1\}$ (onto which the ferrofluid domain is mapped for our analysis). Let $\tilde{f}_m : B_1(\mathbf{0}) \times \mathbb{R} \to \mathbb{C}$ be a function with the property that

$$\tilde{f}_m(r\cos\theta, r\sin\theta, z) = \mathrm{e}^{\mathrm{i}m\theta} f_m(r,z), \qquad r \in [0,1),\ \theta \in \mathbb{T}^1,\ z \in \mathbb{R}, \tag{33}$$

for some $m \in \mathbb{Z}$ and some $f_m : [0,1) \times \mathbb{R} \to \mathbb{C}$ with $f_m(0,z) = 0$ for $m \neq 0$. We refer to such functions as *mode m functions*, such that axisymmetric functions are mode 0 functions.

Remarks 1.4

(i) The radial coefficient $f_0(r,z)$ of a mode 0 function $\tilde{f}_0(x,y,z)$ obviously satisfies $f_0(0,z) = \tilde{f}_0(\mathbf{0},z)$. The same is true for $m \neq 0$ since $f_m(0,z) = 0$ implies that $\tilde{f}_m(\mathbf{0},z) = 0$.

(ii) The radial coefficient $f_m(r,z)$ of the mode m function $\tilde{f}_m(x,y,z)$ is also the radial coefficient of the mode $-m$ function $\tilde{f}_m(x,-y,z)$.

It is convenient to study mode m functions using the Wirtinger-type complex differential operators

$$\partial_\tau := \tfrac{1}{\sqrt{2}}(\partial_x - \mathrm{i}\partial_y), \qquad \partial_{\bar{\tau}} := \tfrac{1}{\sqrt{2}}(\partial_x + \mathrm{i}\partial_y)$$

in place of the Cartesian differential operators ∂_x, ∂_y. Let $\tilde{f}_m : B_1(\mathbf{0}) \times \mathbb{R} \to \mathbb{C}$ be a mode m function with radial coefficient $f_m : [0,1) \times \mathbb{R} \to \mathbb{C}$. It follows that

$$\partial_\tau \tilde{f}_m = \mathrm{e}^{\mathrm{i}(m-1)\theta}\tfrac{1}{\sqrt{2}}\mathcal{D}_m f_m, \qquad \partial_{\bar{\tau}} \tilde{f}_m = \mathrm{e}^{\mathrm{i}(m+1)\theta}\tfrac{1}{\sqrt{2}}\mathcal{D}_{-m} f_m,$$

where $\mathcal{D}_j$ is the Bessel operator

$$\mathcal{D}_j := r^{-j}\frac{\mathrm{d}}{\mathrm{d}r}r^j = \frac{\mathrm{d}}{\mathrm{d}r} + \frac{j}{r}.$$

According to this calculation the operators ∂_τ and $\partial_{\bar{\tau}}$ map a mode m function with radial coefficient f_m to a mode $m-1$ function with radial coefficient $\mathcal{D}_m f_m$ and a mode $m+1$ function with radial coefficient $\mathcal{D}_{-m} f_m$ respectively. Correspondingly, one finds that $\mathcal{D}_m$ and $\mathcal{D}_{-m}$ map a mode m radial coefficient to a mode $m-1$ and a mode $m+1$ radial coefficient respectively, as illustrated diagrammatically in Figure 6 (which commutes). Note that it is actually not necessary to distinguish between mode m and mode $-m$ radial coefficients since the radial coefficient of the mode m function $\tilde{f}_m(x,y)$ is also the radial coefficient of the mode $-m$ function $\tilde{f}_m(x,-y)$ (which explains the apparent ambiguity in this interpretation of $\mathcal{D}_0$.)

We denote the (closed) subspace of the standard Sobolev space

$$H^q(B_1(\mathbf{0}) \times \mathbb{R}; \mathbb{C}) = \left\{ \tilde{f} : B_1(\mathbf{0}) \times \mathbb{R} \to \mathbb{C} \;\middle|\; \|\tilde{f}\|_{H^q}^2 := \sum_{p=0}^{q}\sum_{n=0}^{p}\sum_{i=0}^{n}\binom{n}{i}\ \|\partial_{\bar{\tau}}^{n-i}\partial_\tau^i\partial_z^{p-n}\tilde{f}\|_{L^2}^2 < \infty \right\}$$

consisting of mode m functions by $\tilde{H}^q_{(m)}(B_1(\mathbf{0}) \times \mathbb{R}; \mathbb{C})$. Observe that a mode m function $\tilde{f}_m$ belongs to $L^2(B_1(\mathbf{0}) \times \mathbb{R}; \mathbb{C})$ if and only if its radial coefficient f_m belongs to

$$L_1^2((0,1) \times \mathbb{R}; \mathbb{C}) = \left\{ f : [0,1) \times \mathbb{R} \to \mathbb{C} \;\middle|\; \|f\|_{L_1^2}^2 := 2\pi \int_{\mathbb{R}}\int_0^1 |f(r,z)|^2 r \,\mathrm{d}r\,\mathrm{d}z < \infty \right\},$$

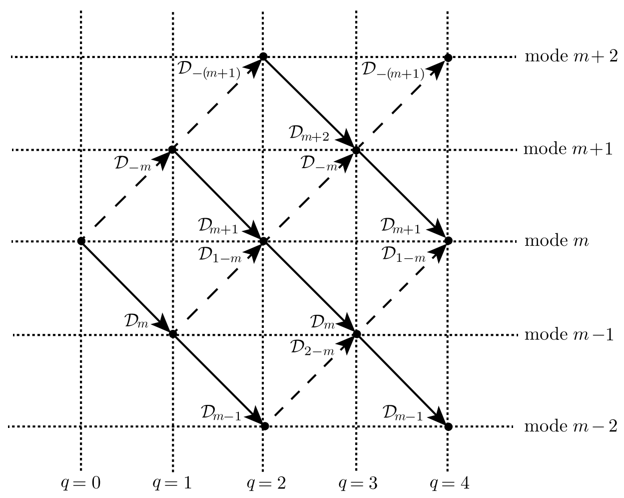

Figure 6: Actions of the Bessel operators. The mode $m+j$ function in column q is $\mathrm{e}^{\mathrm{i}(m+j)\theta}\mathcal{D}_{m_q}\mathcal{D}_{m_{q-1}}\dots\mathcal{D}_{m_1}f_m$, where the indices $\{m_i\}_{i=1}^q$ satisfy $m_i = \pm(m_{i-1}-1)$, with $m_1 = \pm m$, and consist of $\frac{1}{2}(q-j)$ positive and $\frac{1}{2}(q+j)$ non-positive terms.

and that

$$\partial_{\bar{\tau}}^{n-i}\partial_\tau^i \tilde{f}_m = \mathrm{e}^{\mathrm{i}(m+n-2i)\theta}\, 2^{-\frac{n}{2}}\mathcal{D}_{-m+i}^{n-i}\mathcal{D}_m^i f_m, \qquad i = 0,\dots n,$$

where

$$\mathcal{D}_j^i := r^{-j+i}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^i r^j = \mathcal{D}_{j-(i-1)}\mathcal{D}_{j-(i-2)}\dots\mathcal{D}_{j-1}\mathcal{D}_j.$$

One indeed finds that $\tilde{f}_m$ belongs to $\tilde{H}^q_{(m)}(B_1(\mathbf{0})\times\mathbb{R};\mathbb{C})$ if and only if its radial coefficient f_m belongs to

$$H^q_{(m)}((0,1)\times\mathbb{R};\mathbb{C}) = \left\{ f_m : [0,1)\times\mathbb{R}\to\mathbb{C} \;\middle|\; \|f_m\|^2_{H^q_{(m)}} := \sum_{p=0}^q\sum_{n=0}^p 2^{-n}\sum_{i=0}^n \binom{n}{i}\|\mathcal{D}_{-m+i}^{n-i}\mathcal{D}_m^i\partial_z^{p-n}f_m\|^2_{L^2_1} < \infty\right\},$$

and that the mapping $f_m \mapsto \tilde{f}_m$ is an isometric isomorphism (see Groves & Hill [11, §3] for a more precise statement and a discussion of the properties of these function spaces).

2 Analyticity

2.1 *The operator K*

In this section we study the operator K given by (32). Denoting the radial coefficient of $\tilde{\phi}$ by ϕ, such that

$$\tilde{\phi}(x,y,z) = \phi(r,z),$$

we can equivalently define

$$K(\eta)\xi = -(\phi|_{r=1+\eta})_z,$$

where ϕ is the solution of the boundary-value problem

$$\mathcal{D}_1\mathcal{D}_0\phi + \phi_{zz} = 0, \qquad 0 < r < 1+\eta, \tag{34}$$

$$(1+\eta)(\mathcal{D}_0\phi - \eta_z\phi_z) = \xi_z, \qquad r = 1+\eta \tag{35}$$

(which is unique up to additive constants).

The 'flattening' transformation

$$r' = \frac{r}{1+\eta}, \qquad u(r', z) = \phi(r, z)$$

transforms S_1 into the fixed strip $\Sigma = (0,1) \times \mathbb{R}$ and the boundary-value problem (34), (35) into

$$\mathcal{D}_1\mathcal{D}_0 u + u_{zz} = \mathcal{D}_1 F_1(\eta, u) + \partial_z F_2(\eta, u), \qquad 0 < r < 1, \tag{36}$$

$$\mathcal{D}_0 u = F_1(\eta, u) + \xi_z, \qquad r = 1, \tag{37}$$

where we have dropped the primes for notational simplicity and

$$F_1(\eta, u) = r(1+\eta)\eta_z u_z - r^2\eta_z^2 \mathcal{D}_0 u, \qquad F_2(\eta, u) = r(1+\eta)\eta_z \mathcal{D}_0 u - \eta(\eta+2)u_z, \tag{38}$$

so that

$$K(\eta)\xi = -u_z|_{r=1}.$$

This boundary-value problem can be cast as the integral equation

$$u = S(F_1(\eta, u), F_2(\eta, u), \xi), \tag{39}$$

where

$$S(F_1, F_2, \xi) = \mathcal{F}^{-1}\left[\int_0^1 \left(\mathrm{i}kG(r,\tilde{r})\hat{F}_2 - \tilde{\mathcal{D}}_0 G(r,\tilde{r})\hat{F}_1\right)\tilde{r}\,\mathrm{d}\tilde{r} - \mathrm{i}kG(r,1)\hat{\xi}\right]$$

with

$$G(r,\tilde{r}) = \begin{cases} -I_0(|k|r)\left(K_0(|k|\tilde{r}) + \dfrac{K_1(|k|)}{I_1(|k|)}I_0(|k|\tilde{r})\right), & 0 \leq r < \tilde{r}, \\[2ex] -I_0(|k|\tilde{r})\left(K_0(|k|r) + \dfrac{K_1(|k|)}{I_1(|k|)}I_0(|k|r)\right), & \tilde{r} < r < 1. \end{cases}$$

We study (39) for $\eta \in \mathcal{Z}$, $\xi \in H^{3/2}(\mathbb{R})$ and $u \in H^\star(\Sigma)$, where

$$H^\star(\Sigma) := H^2_{(0)}(\Sigma)/\mathbb{R}$$

with norm

$$\|u\|_\star^2 := \|u_z\|^2_{H^1_{(0)}} + \|\mathcal{D}_0 u\|^2_{H^1_{(1)}}.$$

The following result is proved in Section 2.2 below.

Theorem 2.1 *The solution operator S satisfies*

$$\|S(F_1, F_2, \xi)\|_\star \lesssim \|F_1\|_{H^1_{(1)}} + \|F_2\|_{H^1_{(0)}} + \|\xi\|_{3/2}$$

for all $F_1 \in H^1_{(1)}(\Sigma)$, $F_2 \in H^1_{(0)}(\Sigma)$ and $\xi \in H^{3/2}(\mathbb{R})$.

Lemma 2.2 *The formulae* (38) *define analytic functions $F_1 : \mathcal{Z} \times H^\star(\Sigma) \to H^1_{(1)}(\Sigma)$, $F_2 : \mathcal{Z} \times H^\star(\Sigma) \to H^1_{(0)}(\Sigma)$.*

Proof. Clearly

$$\|r\eta_z u_z\|_{L^2_1} \lesssim \|\eta_z\|_\infty \|u_z\|_{L^2_1} \lesssim \|\eta\|_{\mathcal{Z}}\|u\|_\star,$$

(see (31)). Using the calculations

$$\mathcal{D}_1(r\eta_z u_z) = r\eta_z\mathcal{D}_0(u_z) + 2\eta_z u_z, \qquad \begin{aligned}\partial_z(r\eta_z u_z) &= r\eta_{zz}u_z + r\eta_z u_{zz} \\ &= r\eta_{1zz}u_z + r\eta_{2zz}u_z + r\eta_z u_{zz},\end{aligned}$$

we similarly find that

$$\|\mathcal{D}_1(r\eta_z u_z)\|_{L^2_1} \lesssim \|\eta_z\|_\infty(\|\mathcal{D}_0 u_z\|_{L^2_1} + \|u_z\|_{L^2_1}) \lesssim \|\eta\|_{\mathcal{Z}}\|u\|_\star$$

and

$$\|\partial_z(r\eta_z u_z)\|_{L^2_1} \lesssim \|\eta_{1zz}\|_\infty\|u_z\|_{L^2_1} + \|\eta_z\|_\infty\|u_{zz}\|_{L^2_1} + \|\eta_{2zz}u_z\|_{L^2_1} \lesssim \|\eta\|_{\mathcal{Z}}\|u\|_\star + \|\eta_{2zz}u_z\|_{L^2_1}$$

with

$$\begin{aligned}
\|\eta_{2zz}u_z\|_{L^2_1}^2 &\leq \|\eta_{2zz}\|_0^2 \sup_{z\in\mathbb{R}} \|r^{1/2}u_z\|_{L^2(0,1)}^2 \\
&\lesssim \|\eta_{2zz}\|_0^2 \|r^{1/2}u_z(r,z)\|_{H^1(\mathbb{R},L^2(0,1))}^2 \\
&= \|\eta_{2zz}\|_0^2(\|u_z\|_{L^2_1}^2 + \|u_{zz}\|_{L^2_1}^2) \\
&\leq \|\eta\|_{\mathcal{Z}}^2 \|u\|_\star^2.
\end{aligned}$$

It follows that $(\eta,u) \mapsto r\eta_z u_z$ is an analytic mapping $\mathcal{Z} \times H^\star(\Sigma) \to H^1_{(1)}(\Sigma)$, and similar arguments show that $(\eta,u) \mapsto r\eta\eta_z u_z$, $(\eta,u) \mapsto r^2\eta_z^2\mathcal{D}_0 u$ are analytic mappings $\mathcal{Z} \times H^\star(\Sigma) \to H^1_{(1)}(\Sigma)$, such that $F_1 : \mathcal{Z} \times H^\star(\Sigma) \to H^1_{(1)}(\Sigma)$ is analytic. The same method shows that $F_2 : \mathcal{Z} \times H^\star(\Sigma) \to H^1_{(0)}(\Sigma)$ is analytic. □

Theorem 2.3 *For each $\xi \in H^{3/2}(\mathbb{R})$ and each sufficiently small $\eta \in \mathcal{Z}$ the boundary-value problem* (36)*,* (37) *admits a unique solution $u \in H^\star(\Sigma)$. Furthermore, the mapping $\mathcal{Z} \mapsto \mathcal{L}(H^{3/2}(\mathbb{R}), H^\star(\Sigma))$ is analytic at the origin.*

Proof. Define a mapping $T : H^\star(\Sigma) \times \mathcal{Z} \times H^{3/2}(\mathbb{R}) \to H^\star(\Sigma)$ by

$$T(u,\eta,\xi) = u - S(F_1(\eta,u), F_2(\eta,u), \xi),$$

such that the solutions to (39) are precisely the zeros of $T(\cdot,\eta,\xi)$. It follows from Theorem 2.1 and Lemma 2.2 that T is analytic at the origin. Furthermore $T(0,0,0) = 0$ and $\mathrm{d}_1T[0,0,0] = I$ is an isomorphism. It follows from the analytic implicit-function theorem (see Buffoni & Toland [6, Theorem 4.5.4]) that there exists open neighbourhoods $N_1 \subseteq \mathcal{Z}$, $N_2 \subseteq H^{3/2}(\mathbb{R})$ and $N_3 \subseteq H^\star(\Sigma)$ of the origin and analytic function $v : N_1 \times N_2 \to N_3$ such that

$$T(v(\eta,\xi),\eta,\xi) = 0;$$

furthermore $u = v(\eta,\xi)$ for all $(\eta,\xi,u) \in N_1 \times N_2 \times N_3$ with $T(u,\eta,\xi) = 0$. Since u is linear in ξ we can choose $N_2 = H^{3/2}(\mathbb{R})$. □

Corollary 2.4 *The mapping $K : \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ is analytic at the origin.*

Proof. This assertion follows from the formula $K(\eta)\xi = -u_z|_{r=1}$, the analyticity of $u : N_1 \times H^{3/2}(\mathbb{R}) \to H^\star(\Sigma)$ and the facts that $\partial_z : H^\star(\Sigma) \to H^1_{(0)}(\Sigma)$ and $u \mapsto u|_{r=1}$, $H^1_{(0)}(\Sigma) \to H^{1/2}(\mathbb{R})$ are continuous linear operators (see Groves & Hill [11, Lemma 3.24]). □

According to Corollary 2.4 we can choose M sufficiently small and study the equation

$$\mathcal{K}(\eta) - c_0^2(1-\varepsilon^2)\mathcal{L}(\eta) = 0$$

in the set

$$U = \{\eta \in H^2(\mathbb{R}) : \|\eta\|_{\mathcal{Z}} < M\}, \tag{40}$$

noting that $H^2(\mathbb{R})$ is continuously embedded in $\mathcal{Z}$ and U is an open neighbourhood of the origin in $H^2(\mathbb{R})$.

Corollary 2.5 *The formulae* (16)*,* (17) *define analytic functions $U \to L^2(\mathbb{R})$.*

Proof. This observation follows from the formulae

$$\begin{aligned}
\mathcal{K}(\eta) &= -\gamma\left(\nu\left(\frac{1}{1+\eta}\right) - \nu(1)\right) + \left(\frac{1}{1+\eta} - 1\right)\left(\frac{1}{(1+\eta_z^2)^{1/2}} - 1\right) + \frac{1}{1+\eta} - 1 \\
&\quad + \frac{1}{(1+\eta_z^2)^{1/2}} - 1 - \left(\frac{1}{(1+\eta_z^2)^{3/2}} - 1\right)\eta_{zz} - \eta_{zz}, \\
\mathcal{L}(\eta) &= -\tfrac{1}{2}(K(\eta)\eta)^2 - \tfrac{1}{2}K(\eta)\eta K(\eta)\eta^2 - \tfrac{1}{8}(K(\eta)\eta^2)^2 + \frac{\eta_z^2}{2(1+\eta_z^2)} + \frac{\eta_z^2}{2(1+\eta_z^2)}(K(\eta)\eta)^2 \\
&\quad + \frac{\eta_z^2}{8(1+\eta_z^2)}(K(\eta)\eta^2)^2 - \frac{\eta_z^2}{1+\eta_z^2}K(\eta)\eta - \frac{\eta_z^2}{2(1+\eta_z^2)}K(\eta)\eta^2 \\
&\quad + \frac{\eta_z^2}{2(1+\eta_z^2)}K(\eta)\eta K(\eta)\eta^2 + K(\eta)\eta + \tfrac{1}{2}K(\eta)\eta^2
\end{aligned}$$

and

(i) Corollary 2.4,

(i) the fact that the functions

$$\rho \mapsto \nu\left(\frac{1}{1+\rho}\right) - \nu(1), \qquad \rho \mapsto \frac{1}{1+\rho} - 1, \qquad \rho \mapsto \frac{1}{(1+\rho^2)^{1/2}} - 1, \qquad \rho \mapsto \frac{\rho^2}{(1+\rho^2)^{1/2}}$$

are analytic at the origin $H^1(\mathbb{R}) \to H^1(\mathbb{R})$,

(ii) the continuity of the multiplication map $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R})$, $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \to L^2(\mathbb{R})$ (see Hörmander [14, Theorem 8.3.1]),

(iii) the continuity of the embeddings $H^2(\mathbb{R}) \subseteq H^{3/2}(\mathbb{R}) \subseteq H^1(\mathbb{R}) \subseteq H^{1/2}(\mathbb{R}) \subseteq L^2(\mathbb{R})$,

(iv) the fact that $H^{3/2}(\mathbb{R})$ is a Banach algebra. □

2.2 *The linear boundary-value problem*

In this section we prove Theorem 2.1 by estimating the operators

$$\begin{aligned}
\mathcal{G}_1(F) &:= \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}k\tilde{r}G(r,\tilde{r})\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right], \\
\mathcal{G}_2(F) &:= \mathcal{F}^{-1}\left[\int_0^1 -\tilde{\mathcal{D}}_0 G(r,\tilde{r})\tilde{r}\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right], \\
\mathcal{G}_3(\xi) &:= \mathcal{F}^{-1}[-\mathrm{i}kG(r,1)\hat{\xi}] = \mathcal{F}^{-1}\left[-\mathrm{i}\frac{I_0(|k|r)}{I_1(|k|)}\hat{\xi}\right]
\end{aligned}$$

in Lemmata 2.10–2.12 below. For this purpose we introduce the functions

$$H_1(r,\tilde{r}) = \begin{cases} -|k|I_1(|k|r)\left(K_0(|k|\tilde{r}) + \dfrac{K_1(|k|)}{I_1(|k|)}I_0(|k|\tilde{r})\right), & 0 \le r < \tilde{r}, \\[2ex] |k|I_0(|k|\tilde{r})\left(K_1(|k|r) - \dfrac{K_1(|k|)}{I_1(|k|)}I_1(|k|r)\right), & \tilde{r} < r < 1, \end{cases}$$

$$H_2(r,\tilde{r}) = \begin{cases} |k|I_0(|k|r)\left(K_1(|k|\tilde{r}) - \dfrac{K_1(|k|)}{I_1(|k|)}I_1(|k|\tilde{r})\right), & 0 \le r < \tilde{r}, \\[2ex] -|k|I_1(|k|\tilde{r})\left(K_0(|k|r) + \dfrac{K_1(|k|)}{I_1(|k|)}I_0(|k|r)\right), & \tilde{r} < r < 1, \end{cases}$$

$$H_3(r,\tilde{r}) = \begin{cases} |k|^2I_1(|k|r)\left(K_1(|k|\tilde{r}) - \dfrac{K_1(|k|)}{I_1(|k|)}I_1(|k|\tilde{r})\right), & 0 \le r < \tilde{r}, \\[2ex] |k|^2I_1(|k|\tilde{r})\left(K_1(|k|r) - \dfrac{K_1(|k|)}{I_1(|k|)}I_1(|k|r)\right), & \tilde{r} < r < 1, \end{cases}$$

which are the formal derivatives $G_r(r,\tilde{r})$, $G_{\tilde{r}}(r,\tilde{r})$ and $G_{r\tilde{r}}(r,\tilde{r})$ of $G(r,\tilde{r})$ respectively, and establish the following auxiliary results.

Proposition 2.6 *The function $G(r,\tilde{r})$ satisfies*

$$\int_0^1 \tilde{r}|G(r,\tilde{r})|\,\mathrm{d}\tilde{r} = \frac{1}{|k|^2}, \qquad \int_0^1 r|G(r,\tilde{r})|\,\mathrm{d}r = \frac{1}{|k|^2}$$

for all $k \in \mathbb{R}$.

Proof. We find that

$$\begin{aligned}\int_0^1 r|G(r,\tilde r)|\,\mathrm{d}r &= \left(K_0(|k|\tilde r)+\frac{K_1(|k|)}{I_1(|k|)}I_0(|k|\tilde r)\right)\int_0^{\tilde r} rI_0(|k|r)\,\mathrm{d}r\\ &\qquad + I_0(|k|\tilde r)\int_{\tilde r}^1 r\left(K_0(|k|r)+\frac{K_1(|k|)}{I_1(|k|)}I_0(|k|r)\right)\mathrm{d}r\\ &= \frac{1}{|k|^2}\end{aligned}$$

and

$$\int_0^1 \tilde r|G(r,\tilde r)|\,\mathrm{d}\tilde r = \int_0^1 r|G(r,\tilde r)|\,\mathrm{d}r = \frac{1}{|k|^2}. \qquad \square$$

Proposition 2.7 *The function $H_1(r,\tilde r)$ satisfies*

$$\int_0^1 \tilde r|H_1(r,\tilde r)|\,\mathrm{d}\tilde r \lesssim \frac{1}{|k|}, \qquad \int_0^1 r|H_1(r,\tilde r)|\,\mathrm{d}r \lesssim \frac{1}{|k|}$$

for all $k\in\mathbb{R}$.

Proof. We note that

$$2|k|rI_1(|k|r)K_1(|k|r) \to \begin{cases} 0 & \text{as } |k|r\to 0,\\ 1 & \text{as } |k|r\to\infty\end{cases}$$

and

$$2|k|rI_1(|k|r)^2\frac{K_1(|k|)}{I_1(|k|)} \le 2|k|I_1(|k|)K_1(|k|) \to \begin{cases} 0 & \text{as } |k|\to 0,\\ 1 & \text{as } |k|\to\infty,\end{cases}$$

so that these quantities are bounded over $r\in[0,1]$ and $k\in\mathbb{R}$. We therefore find that

$$\begin{aligned}|k|\int_0^1 \tilde r|H_1(r,\tilde r)\,\mathrm{d}\tilde r &= |k|\left(K_1(|k|r)-\frac{K_1(|k|)}{I_1(|k|)}I_1(|k|r)\right)\int_0^r |k|\tilde r I_0(|k|\tilde r)\,\mathrm{d}\tilde r\\ &\qquad + |k|I_1(|k|r)\int_r^1 |k|\tilde r\left(K_0(|k|\tilde r)+\frac{K_1(|k|)}{I_1(|k|)}I_0(|k|\tilde r)\right)\mathrm{d}\tilde r\\ &= -2|k|rI_1(|k|r)^2\frac{K_1(|k|)}{I_1(|k|)}+2|k|rI_1(|k|r)K_1(|k|r)\\ &\lesssim 1.\end{aligned}$$

Next we record the estimates

$$\begin{aligned}0 &\le \tfrac{\pi}{2}|k|\tilde r I_1(|k|\tilde r)K_0(|k|\tilde r)L_0(|k|\tilde r)-\tfrac{\pi}{2}|k|\tilde r I_0(|k|\tilde r)K_1(|k|\tilde r)L_0(|k|\tilde r)\\ &\qquad -\pi|k|\tilde r I_0(|k|\tilde r)K_0(|k|\tilde r)L_1(|k|\tilde r)+\tfrac{\pi}{2}I_0(|k|\tilde r)\\ &\to \begin{cases} \frac{\pi}{2} & \text{as } |k|\tilde r\to 0,\\ 1 & \text{as } |k|\tilde r\to\infty,\end{cases}\end{aligned}$$

$$0\le \tfrac{\pi}{2}I_0(|k|\tilde r)\left(1-\frac{L_1(|k|)}{I_1(|k|)}\right)\le \tfrac{\pi}{2}I_0(|k|)\left(1-\frac{L_1(|k|)}{I_1(|k|)}\right)\to\begin{cases}\frac{\pi}{2} & \text{as } |k|\to 0,\\ 1 & \text{as } |k|\to\infty,\end{cases}$$

where L_ν is the modified Struve function of the first kind and order ν. Using the fact that $h: s\mapsto \pi s(I_1(s)L_0(s)-I_0(s)L_1(s))$ is increasing (since $h'(s)=2sI_1(s)>0$ for $s>0$ with $h'(0)=0$), we furthermore find that

$$
\begin{aligned}
0 &\le \pi|k|\tilde r I_0(|k|\tilde r)\Big(I_1(|k|\tilde r)L_0(|k|\tilde r) - I_0(|k|\tilde r)L_1(|k|\tilde r)\Big)\frac{K_1(|k|\tilde r)}{I_1(|k|\tilde r)}\\
&\le \pi|k|I_0(|k|)\Big(I_1(|k|)L_0(|k|) - I_0(|k|)L_1(|k|)\Big)\frac{K_1(|k|)}{I_1(|k|)}\\
&\to \left\{\begin{array}{ll} 0 & \text{as } |k| \to 0,\\ 1 & \text{as } |k| \to \infty.\end{array}\right.
\end{aligned}
$$

Using these estimates we conclude that

$$
\begin{aligned}
|k|\int_0^1 r|H_1(r,\tilde r)|\,\mathrm{d}r &= |k|\left(K_0(|k|\tilde r) + \frac{K_1(|k|)}{I_1(|k|)}I_0(|k|\tilde r)\right)\int_0^{\tilde r}|k|rI_1(|k|r)\,\mathrm{d}\tilde r\\
&\qquad + |k|I_0(|k|\tilde r)\int_{\tilde r}^1 |k|r\left(K_1(|k|r) - \frac{K_1(|k|)}{I_1(|k|)}I_1(|k|r)\right)\mathrm{d}\tilde r\\
&= \tfrac{\pi}{2}|k|\tilde r I_1(|k|\tilde r)K_0(|k|\tilde r)L_0(|k|\tilde r) - \tfrac{\pi}{2}|k|\tilde r I_0(|k|\tilde r)K_1(|k|\tilde r)L_0(|k|\tilde r)\\
&\qquad - \pi|k|\tilde r I_0(|k|\tilde r)K_0(|k|\tilde r)L_1(|k|\tilde r) + \tfrac{\pi}{2}I_0(|k|\tilde r)\frac{L_1(|k|)}{I_1(|k|)}\\
&\qquad + \pi|k|\tilde r I_0(|k|\tilde r)\Big(I_1(|k|\tilde r)L_0(|k|\tilde r) - I_0(|k|\tilde r)L_1(|k|\tilde r)\Big)\frac{K_1(|k|)}{I_1(|k|)}\\
&\lesssim 1.
\end{aligned}
$$

□

Corollary 2.8 *The function $H_2(r,\tilde r)$ satisfies*

$$
\int_0^1 \tilde r|H_2(r,\tilde r)|\,\mathrm{d}\tilde r \lesssim \frac{1}{|k|}, \qquad \int_0^1 r|H_2(r,\tilde r)|\,\mathrm{d}r \lesssim \frac{1}{|k|}
$$

for all $k\in\mathbb{R}$.

Proposition 2.9 *The function $H_3(r,\tilde r)$ satisfies*

$$
\int_0^1 \tilde r|H_3(r,\tilde r)|\,\mathrm{d}\tilde r \lesssim 1, \qquad \int_0^1 r|H_3(r,\tilde r)|\,\mathrm{d}r \lesssim 1
$$

for all $k\in\mathbb{R}$.

Proof. Using the estimates

$$
0 \le \tfrac{\pi}{2}I_1(|k|r)\left(1 - \frac{L_1(|k|)}{I_1(|k|)}\right) \le \tfrac{\pi}{2}I_1(|k|)\left(1 - \frac{L_1(|k|)}{I_1(|k|)}\right) \to \left\{\begin{array}{ll} 0 & \text{as } |k| \to 0,\\ 1 & \text{as } |k| \to \infty,\end{array}\right.
$$

$$
0 \le \tfrac{\pi}{2}\big(I_1(|k|r) - L_1(|k|r)\big) \to \left\{\begin{array}{ll} 0 & \text{as } |k| \to 0,\\ 1 & \text{as } |k| \to \infty,\end{array}\right.
$$

we find that

$$
\begin{aligned}
\int_0^1 \tilde r|H_3(r,\tilde r)|\,\mathrm{d}\tilde r &= |k|\left(K_1(|k|r) - \frac{K_1(|k|)}{I_1(|k|)}I_1(|k|r)\right)\int_0^r |k|\tilde r I_1(|k|\tilde r)\,\mathrm{d}\tilde r\\
&\qquad + |k|I_1(|k|r)\int_r^1 |k|\tilde r\left(K_1(|k|\tilde r) - \frac{K_1(|k|)}{I_1(|k|)}I_1(|k|\tilde r)\right)\mathrm{d}\tilde r\\
&= \tfrac{\pi}{2}I_1(|k|r)\frac{L_1(|k|)}{I_1(|k|)} - \tfrac{\pi}{2}L_1(|k|r)\\
&\lesssim 1
\end{aligned}
$$

and

$$\int_0^1 r|H_3(r,\tilde{r})|\,\mathrm{d}r = \int_0^1 \tilde{r}|H_3(r,\tilde{r})|\,\mathrm{d}\tilde{r} \lesssim 1. \qquad \square$$

Lemma 2.10 *The estimate*

$$\|\mathcal{G}_1(F)\|_\star \lesssim \|F\|_{H^1_{(0)}}$$

holds for all $F \in H^1_{(0)}(\Sigma)$.

Proof. It follows fom Proposition 2.6 that

$$\begin{aligned}
\|\partial_z \mathcal{G}_1(F)\|^2_{L^2_1} &\lesssim \int_{-\infty}^{\infty}\int_0^1 r\left|\int_0^1 -|k|^2\tilde{r}G(r,\tilde{r})\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right|^2\,\mathrm{d}r\,\mathrm{d}k\\
&\lesssim \int_{-\infty}^{\infty}\int_0^1 r\int_0^1 |k|^2\tilde{r}|G(r,\tilde{r})|\,\mathrm{d}\tilde{r}\int_0^1 |k|^2\tilde{r}|G(r,\tilde{r})||\hat{F}(\tilde{r})|^2\,\mathrm{d}\tilde{r}\,\mathrm{d}r\,\mathrm{d}k\\
&\lesssim \int_{-\infty}^{\infty}\int_0^1 \left(\int_0^1 |k|^2 r|G(r,\tilde{r})|\,\mathrm{d}r\right)\tilde{r}|\hat{F}(\tilde{r})|^2\,\mathrm{d}\tilde{r}\,\mathrm{d}k\\
&\lesssim \|F\|^2_{L^2_1}
\end{aligned}$$

and hence

$$\|\partial_z^2\mathcal{G}_1(F)\|_{L^2_1} = \|\partial_z\mathcal{G}_1(F_z)\|_{L^2_1} \lesssim \|F_z\|_{L^2_1}.$$

Next we note that

$$\mathcal{D}_0\mathcal{G}_1(F) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}k\tilde{r}H_1(r,\tilde{r})\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right]$$

and using Proposition 2.7 that

$$\begin{aligned}
\|\mathcal{D}_0\mathcal{G}_1(F)\|^2_{L^2_1} &\lesssim \int_{-\infty}^{\infty}\int_0^1 r\left|\int_0^1 \mathrm{i}k\tilde{r}H_1(r,\tilde{r})\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right|^2\,\mathrm{d}r\,\mathrm{d}k\\
&\lesssim \int_{-\infty}^{\infty}\int_0^1 r\int_0^1 |k|\tilde{r}|H_1(r,\tilde{r})|\,\mathrm{d}\tilde{r}\int_0^1 |k|\tilde{r}|H_1(r,\tilde{r})||\hat{F}(\tilde{r})|^2\,\mathrm{d}\tilde{r}\,\mathrm{d}r\,\mathrm{d}k\\
&\lesssim \int_{-\infty}^{\infty}\int_0^1 \left(\int_0^1 |k|r|H_1(r,\tilde{r})|\,\mathrm{d}r\right)\tilde{r}|\hat{F}(\tilde{r})|^2\,\mathrm{d}\tilde{r}\,\mathrm{d}k\\
&\lesssim \|F\|^2_{L^2_1}
\end{aligned}$$

and hence

$$\|\partial_z\mathcal{D}_0\mathcal{G}_1(F)\|_{L^2_1} = \|\mathcal{D}_0\mathcal{G}_1(F_z)\|_{L^2_1} \lesssim \|F_z\|_{L^2_1}.$$

Finally, using the identity

$$\mathcal{D}_1\mathcal{D}_0\mathcal{G}_1(F) + \partial_z^2\mathcal{G}_1(F) = F_z,$$

which follows from the definition of $\mathcal{G}_1$, we find that

$$\|\mathcal{D}_1\mathcal{D}_0\mathcal{G}_1(F)\|_{L^2_1} \leq \|\partial_z^2\mathcal{G}_1(F)\|_{L^2_1} + \|F_z\|_{L^2_1} \lesssim \|F_z\|_{L^2_1}. \qquad \square.$$

Lemma 2.11 *The estimate*

$$\|\mathcal{G}_2(F)\|_\star \lesssim \|F\|_{H^1_{(1)}}$$

holds for all $F \in H^1_{(1)}(\Sigma)$.

Proof. Since

$$\partial_z\mathcal{G}_2(F) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}k\tilde{r}H_2(r,\tilde{r})\hat{F}(\tilde{r})\,\mathrm{d}\tilde{r}\right]$$

it follows from Corollary 2.8 that

$$
\begin{aligned}
\|\partial_z \mathcal{G}_2(F)\|_{L_1^2}^2 &\lesssim \int_{-\infty}^{\infty} \int_0^1 r \left| \int_0^1 \mathrm{i}k\tilde{r} H_2(r,\tilde{r}) \hat{F}(\tilde{r}) \,\mathrm{d}\tilde{r} \right|^2 \mathrm{d}r \,\mathrm{d}k \\
&\lesssim \int_{-\infty}^{\infty} \int_0^1 r|k| \int_0^1 \tilde{r}|H_2(r,\tilde{r})| \,\mathrm{d}\tilde{r} \int_0^1 |k|\tilde{r}|H_2(r,\tilde{r})||\hat{F}(\tilde{r})|^2 \,\mathrm{d}\tilde{r}\,\mathrm{d}r\,\mathrm{d}k \\
&\lesssim \int_{-\infty}^{\infty} \int_0^1 \left(\int_0^1 |k|r|H_2(r,\tilde{r})| \,\mathrm{d}r \right) \tilde{r}|\hat{F}(\tilde{r})|^2 \,\mathrm{d}\tilde{r}\,\mathrm{d}k \\
&\lesssim \|F\|_{L_1^2}^2
\end{aligned}
$$

and hence

$$\|\partial_z^2 \mathcal{G}_2(F)\|_{L_1^2} = \|\partial_z \mathcal{G}_2(F_z)\|_{L_1^2} \lesssim \|F_z\|_{L_1^2}.$$

Next we note that

$$\mathcal{D}_0 \mathcal{G}_2(F) = \mathcal{F}^{-1}\left[\int_0^1 \tilde{r} H_3(r,\tilde{r}) \hat{F}(\tilde{r}) \,\mathrm{d}\tilde{r} \right] - F,$$

so that

$$
\begin{aligned}
\|\mathcal{D}_0 \mathcal{G}_2(F)\|_{L_1^2}^2 &\lesssim \int_{-\infty}^{\infty} \int_0^1 r \left| \int_0^1 \tilde{r} H_3(r,\tilde{r}) \hat{F}(\tilde{r}) \,\mathrm{d}\tilde{r} \right|^2 \mathrm{d}r \,\mathrm{d}k + \|F\|_{L_1^2} \\
&\lesssim \int_{-\infty}^{\infty} \int_0^1 r \int_0^1 \tilde{r}|H_3(r,\tilde{r})| \,\mathrm{d}\tilde{r} \int_0^1 \tilde{r}|H_3(r,\tilde{r})||\hat{F}(\tilde{r})|^2 \,\mathrm{d}\tilde{r}\,\mathrm{d}r\,\mathrm{d}k + \|F\|_{L_1^2} \\
&\lesssim \int_{-\infty}^{\infty} \int_0^1 \left(\int_0^1 r|H_3(r,\tilde{r})| \,\mathrm{d}r \right) \tilde{r}|\hat{F}(\tilde{r})|^2 \,\mathrm{d}\tilde{r}\,\mathrm{d}k \\
&\lesssim \|F\|_{L_1^2}^2,
\end{aligned}
$$

where we have used Proposition 2.9, and hence

$$\|\partial_z \mathcal{D}_0 \mathcal{G}_2(F)\|_{L_1^2} = \|\mathcal{D}_0 \mathcal{G}_2(F_z)\|_{L_1^2} \lesssim \|F_z\|_{L_1^2}.$$

Finally, using the identity

$$\mathcal{D}_1 \mathcal{D}_0 \mathcal{G}_2(F) + \partial_z^2 \mathcal{G}_2(F) = \mathcal{D}_1 F,$$

which follows from the definition of $\mathcal{G}_2$, we find that

$$\|\mathcal{D}_1 \mathcal{D}_0 \mathcal{G}_2(F)\|_{L_1^2} \leq \|\partial_z^2 \mathcal{G}_2(F)\|_{L_1^2} + \|\mathcal{D}_1 F\|_{L_1^2} \lesssim \|F_z\|_{L_1^2} + \|\mathcal{D}_1 F\|_{L_1^2}. \qquad \square.$$

Lemma 2.12 *The estimate*

$$\|\mathcal{G}_3(\xi)\|_\star \lesssim \|\xi\|_{3/2}$$

holds for all $\xi \in H^{3/2}(\mathbb{R})$.

Proof. First note that

$$\tfrac{1}{2}|k|^2 \left(\frac{I_0(|k|)^2}{I_1(|k|)^2} - 1 \right) \lesssim (1+|k|^2)^{1/2}$$

since

$$\frac{|k|^2}{2(1+|k|^2)^{1/2}} \left(\frac{I_0(|k|)^2}{I_1(|k|)^2} - 1 \right) \to \begin{cases} 2 & \text{as } |k| \to 0, \\ \frac{1}{2} & \text{as } |k| \to \infty, \end{cases}$$

from which it follows that

$$
\begin{aligned}
\|\partial_z \mathcal{G}_3(\xi)\|_{L_1^2} &= \int_{-\infty}^{\infty} \int_0^1 r|k|^2 \frac{I_0(|k|r)^2}{I_1(|k|)^2} |\hat{\xi}|^2 \,\mathrm{d}r\,\mathrm{d}k \\
&= \int_{-\infty}^{\infty} \int_0^1 \tfrac{1}{2}|k|^2 \left(\frac{I_0(|k|)^2}{I_1(|k|)^2} - 1 \right) |\hat{\xi}|^2 \,\mathrm{d}k \\
&\lesssim \|\xi\|_{1/2}^2
\end{aligned}
$$

and hence that

$$\|\partial_z^2 \mathcal{G}_3(\xi)\|_{L_1^2} = \|\mathcal{G}_3(\xi_z)\|_{L_1^2} \lesssim \|\xi_z\|_{1/2}^2.$$

Similarly

$$\tfrac{1}{2}|k|^2 \left(1 - \frac{I_0(|k|)^2}{I_1(|k|)^2}\right) + |k| \frac{I_0(|k|)}{I_1(|k|)} \lesssim (1+|k|^2)^{1/2}$$

since

$$\frac{|k|^2}{2(1+|k|^2)^{1/2}} \left(1 - \frac{I_0(|k|)^2}{I_1(|k|)^2}\right) + \frac{|k|}{(1+|k|^2)^{1/2}} \frac{I_0(|k|)}{I_1(|k|)} \to \begin{cases} 0 & \text{as } |k| \to 0, \\ \frac{1}{2} & \text{as } |k| \to \infty, \end{cases}$$

from which it follows that

$$\begin{aligned} \|\mathcal{D}_0 \mathcal{G}_3(\xi)\|_{L_1^2} &= \int_{-\infty}^{\infty} \int_0^1 r|k|^2 \frac{I_1(|k|r)^2}{I_1(|k|)^2} |\hat{\xi}|^2 \, \mathrm{d}r \, \mathrm{d}k \\ &= \int_{-\infty}^{\infty} \left(\tfrac{1}{2}|k|^2 \left(1 - \frac{I_0(|k|)^2}{I_1(|k|)^2}\right) + |k| \frac{I_0(|k|)}{I_1(|k|)}\right) |\hat{\xi}|^2 \, \mathrm{d}r \, \mathrm{d}k \\ &\lesssim \|\xi\|_{1/2}^2 \end{aligned}$$

and hence that

$$\|\partial_z \mathcal{D}_0 \mathcal{G}_3(\xi)\|_{L_1^2} = \|\mathcal{D}_0 \mathcal{G}_3(\xi_z)\|_{L_1^2} \lesssim \|\xi_z\|_{1/2}^2.$$

Finally, using the identity

$$\mathcal{D}_1 \mathcal{D}_0 \mathcal{G}_3(\xi) + \partial_z^2 \mathcal{G}_3(\xi) = 0,$$

which follows from the definition of $\mathcal{G}_3$, we find that

$$\|\mathcal{D}_1 \mathcal{D}_0 \mathcal{G}_3(\xi)\|_{L_1^2} \leq \|\partial_z^2 \mathcal{G}_3(\xi)\|_{L_1^2} \lesssim \|\xi_z\|_{1/2}. \qquad \square.$$

2.3 Expansions

Using the results in Section 2.1 above, we obtain the expansions

$$u(\eta, \xi) = \sum_{j=0}^{\infty} u^j(\eta, \xi), \tag{41}$$

where u^j is homogeneous of degree j in η and linear in ξ, and

$$K(\eta) = \sum_{j=0}^{\infty} K_j(\eta), \qquad \mathcal{K}(\eta) = \sum_{j=0}^{\infty} \mathcal{K}_j(\eta), \qquad \mathcal{L}(\eta) = \sum_{j=0}^{\infty} \mathcal{L}_j(\eta),$$

where K_j, $\mathcal{K}_j$, $\mathcal{L}_j$ are homogeneous of degree j in η. Note in particular the formulae

$$\mathcal{K}_1(\eta) = (\gamma - 1)\eta - \eta_{zz}, \tag{42}$$
$$\mathcal{K}_2(\eta) = A_0 \eta^2 - \tfrac{1}{2}\eta_z^2, \tag{43}$$
$$\mathcal{K}_3(\eta) = B_0 \eta^3 + \tfrac{1}{2}\eta\eta_z^2 + \tfrac{3}{2}\eta_z^2 \eta_{zz}, \tag{44}$$

$$\mathcal{L}_1(\eta) = K_0 \eta, \tag{45}$$
$$\mathcal{L}_2(\eta) = \tfrac{1}{2}\big(\eta_z^2 - (K_0\eta)^2 + K_0 \eta^2 + 2K_1(\eta)\eta\big), \tag{46}$$
$$\mathcal{L}_3(\eta) = -\eta_z^2 K_0 \eta - \tfrac{1}{2}(K_0\eta)(K_0\eta^2 + 2K_1(\eta)\eta) + \tfrac{1}{2}K_1(\eta)\eta^2 + K_2(\eta)\eta, \tag{47}$$

where

$$A_0 = -\gamma - \tfrac{1}{2}\gamma\nu''(1) + 1, \qquad B_0 = \gamma + \gamma\nu''(1) + \tfrac{1}{6}\gamma\nu'''(1) - 1,$$

which are obtained from equations (16), (17).

The terms in the expansion of $u(\eta, \xi)$ can be computed by proceeding formally. Substituting (41) into equations (36)–(37) and equating terms which are homogeneous of order j in η yields a boundary-value problem for u^j in terms of $u^0, \ldots, u^{j-1}$. Formulae for the terms in the expansion for $K(\eta)$ in terms of Fourier multipliers are then recovered from the formula

$$K_j(\eta)(\xi) = -u_z^j(\eta, \xi)|_{r=1}.$$

Remark 2.13 *This method leads to formulae involving ever more derivatives of η and ξ in the individual terms in the formulae for $K_j(\eta)$; the overall validity of the formulae arises from subtle cancellations between the terms (see Nicholls and Reitich [15, §2.2] for a discussion of this phenomenon in the context of the classical Dirichlet–Neumann operator).*

Proposition 2.14 *The operator $K_0 \in \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ is given by the formula*

$$K_0\xi = f(D)\xi,$$

which also defines an operator in $\mathcal{L}(H^{s+1}(\mathbb{R}), H^s(\mathbb{R}))$ for each $s \geq 0$.

Proof. The solution to the boundary-value problem

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_0 u^0 + u^0_{zz} &= 0, \qquad 0<r<1,\\ \mathcal{D}_0 u^0 &= \xi_z, \qquad r=1, \end{aligned}$$

for u^0 is

$$u^0 = \mathcal{F}^{-1}\left[\frac{\mathrm{i}kI_0(|k|r)}{|k|I_1(|k|)}\hat{\xi}\right],$$

such that

$$K_0\xi = -(u^0_z)|_{r=1} = f(D)\xi.$$

Furthermore, the estimate

$$f(k) \lesssim \sqrt{1+k^2}, \qquad k \in \mathbb{R},$$

which follows from the calculation

$$\frac{f(k)}{\sqrt{1+k^2}} = \frac{|k|I_0(|k|)}{\sqrt{1+k^2}I_1(|k|)} \to \begin{cases} 2 & \text{as } |k|\to 0,\\ 1 & \text{as } |k|\to\infty, \end{cases}$$

implies that

$$\|K_0\xi\|_s \lesssim \|\xi\|_{s+1}$$

for all $s \geq 0$. □

Proposition 2.15 *The operators K_1 and K_2 are given by the formulae*

$$\begin{aligned} K_1(\eta)\xi &= -(\eta\xi_z)_z - K_0(\eta K_0\xi),\\ K_2(\eta)\xi &= \tfrac{1}{2}(\eta^2K_0\xi)_{zz} + \tfrac{1}{2}K_0(\eta^2\xi_{zz}) + \tfrac{1}{2}(\eta^2\xi_z)_z - \tfrac{1}{2}K_0(\eta^2K_0\xi) + K_0(\eta K_0(\eta K_0\xi)) \end{aligned}$$

for each $\eta \in H^2(\mathbb{R})$ and $\xi \in H^{3/2}(\mathbb{R})$ (see Remark 2.13).

Proof. The solution to the boundary-value problem

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_0 u^1 + u^1_{zz} &= \mathcal{D}_1(r\eta_z u^0_z) + \partial_z(r\eta_z\mathcal{D}_0u^0 - 2\eta u^0_z), \qquad 0<r<1,\\ \mathcal{D}_0 u^1 &= r\eta_z u^0_z, \qquad r=1, \end{aligned}$$

for u^1 is

$$u^1 = r\eta\mathcal{D}_0u^0 + w^1,$$

where

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_0 w^1 + w^1_{zz} &= 0, \qquad 0<r<1,\\ \mathcal{D}_0 w^1 &= (\eta u^0_z)_z, \qquad r=1, \end{aligned}$$

such that

$$u^1 = r\eta\mathcal{D}_0u^0 + \mathcal{F}^{-1}\left[\frac{\mathrm{i}kI_0(|k|r)}{|k|I_1(|k|)}\mathcal{F}[\eta u^0_z|_{r=1}]\right].$$

It follows that

$$K_1(\eta)\xi = -(u_z^1)|_{r=1} = -(\eta\xi_z)_z - K_0(\eta K_0\xi).$$

Similarly, the solution to the boundary-value problem

$$\begin{aligned}
\mathcal{D}_1\mathcal{D}_0 u^2 + u_{zz}^2 &= \mathcal{D}_1(r\eta_z u_z^1 + r\eta\eta_z u_z^0 - r^2\eta_z\mathcal{D}_0 u^0), && 0<r<1,\\
&\quad + \partial_z(r\eta_z\mathcal{D}_0 u^1 + r\eta\eta_z\mathcal{D}_0 u^0 - \eta^2 u_z^0 - 2\eta u_z^1),\\
\mathcal{D}_0 u^2 &= r\eta_z u_z^1 + r\eta\eta_z u_z^0 - r^2\eta_z\mathcal{D}_0 u^0, && r=1,
\end{aligned}$$

for u^2 is

$$u^2 = -\tfrac{1}{2}\eta^2\mathcal{D}_0(r^2\mathcal{D}_0 u^0) + r\eta\mathcal{D}_0 u^1 + w^2,$$

where

$$\begin{aligned}
\mathcal{D}_1\mathcal{D}_0 w^2 + w_{zz}^2 &= 0, && 0<r<1,\\
\mathcal{D}_0 w^2 &= (u_z^1)_z + (\tfrac{1}{2}\eta^2 u_z^0)_z - (\tfrac{1}{2}(\eta^2\mathcal{D}_0 u^0)_z)_z, && r=1,
\end{aligned}$$

such that

$$\begin{aligned}
u^2 = &-\tfrac{1}{2}\eta^2\mathcal{D}_0(r^2\mathcal{D}_0 u^0) + r\eta\mathcal{D}_0 u^1 + \mathcal{F}^{-1}\left[\frac{\mathrm{i}kI_0(|k|r)}{|k|I_1(|k|)}\mathcal{F}[\tfrac{1}{2}\eta^2 u_z^0|_{r=1}]\right]\\
&- \mathcal{F}^{-1}\left[\frac{\mathrm{i}kI_0(|k|r)}{|k|I_1(|k|)}\mathcal{F}[\tfrac{1}{2}(\eta^2\mathcal{D}_0 u^0)_z|_{r=1}]\right] + \mathcal{F}^{-1}\left[\frac{\mathrm{i}kI_0(|k|r)}{|k|I_1(|k|)}\mathcal{F}[\eta u_z^1|_{r=1}]\right].
\end{aligned}$$

We conclude that

$$\begin{aligned}
K_2(\eta)\xi &= -(u_z^2)|_{r=1}\\
&= \tfrac{1}{2}(\eta^2\mathcal{D}_0^2 u^0)_z + (\eta^2\mathcal{D}_0 u^0)_z - (\eta\eta_z u_z^0)_z - \tfrac{1}{2}K_0((\eta^2\xi_z)_z) + \tfrac{1}{2}K_0(\eta^2 u_z^0) - K_0(\eta K_1(\eta)\xi)\\
&= \tfrac{1}{2}(\eta^2 K_0\xi)_{zz} + \tfrac{1}{2}K_0(\eta^2\xi_{zz}) + \tfrac{1}{2}(\eta^2\xi_z)_z - \tfrac{1}{2}K_0(\eta^2 K_0\xi) + K_0(\eta K_0(\eta K_0\xi)).
\end{aligned}$$

□

Corollary 2.16 *The formulae*

$$\begin{aligned}
\mathcal{K}_1(\eta) &= (\gamma-1)\eta - \eta_{zz},\\
\mathcal{K}_2(\eta) &= A_0\eta^2 - \tfrac{1}{2}\eta_z^2,\\
\mathcal{K}_3(\eta) &= B_0\eta^3 + \tfrac{1}{2}\eta\eta_z^2 + \tfrac{3}{2}\eta_z^2\eta_{zz},\\[1em]
\mathcal{L}_1(\eta) &= K_0\eta,\\
\mathcal{L}_2(\eta) &= \tfrac{1}{2}(\eta_z^2 - (K_0\eta)^2 - (\eta^2)_{zz} - 2K_0(\eta K_0\eta) + K_0\eta^2),\\
\mathcal{L}_3(\eta) &= \tfrac{1}{2}(K_0\eta)(\eta^2)_{zz} + (K_0\eta)(K_0(\eta K_0\eta)) - \tfrac{1}{2}(K_0\eta)(K_0\eta^2) - \eta_z^2(K_0\eta)\\
&\quad + \tfrac{1}{2}(\eta^2 K_0\eta)_{zz} + \tfrac{1}{2}K_0(\eta^2\eta_{zz}) - \tfrac{1}{2}(\eta^2\eta_z)_z - \tfrac{1}{2}K_0(\eta^2 K_0\eta)\\
&\quad + K_0(\eta K_0(\eta K_0\eta)) - \tfrac{1}{2}K_0(\eta K_0\eta^2)
\end{aligned}$$

hold for all $\eta \in H^2(\mathbb{R}^2)$ (see Remark 2.13).

Lemma 2.17

(i) The estimates

$$\begin{aligned}
\|\mathcal{K}_2(\eta)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}\|\eta\|_2, &\qquad \|\mathrm{d}\mathcal{K}_2[\eta](\rho)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}\|\rho\|_2,\\
\|\mathcal{L}_2(\eta)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}\|\eta\|_2, &\qquad \|\mathrm{d}\mathcal{L}_2[\eta](\rho)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}\|\rho\|_2
\end{aligned}$$

hold for all η, $\rho \in H^2(\mathbb{R})$.

(ii) The estimates

$$\|\mathcal{K}_3(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_2, \qquad \|\mathrm{d}\mathcal{K}_3[\eta](\rho)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\rho\|_2 + \|\eta\|_{\mathcal{Z}}\|\eta\|_2\|\rho\|_2,$$
$$\|\mathcal{L}_3(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_2, \qquad \|\mathrm{d}\mathcal{L}_3[\eta](\rho)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\rho\|_2 + \|\eta\|_{\mathcal{Z}}\|\eta\|_2\|\rho\|_2$$

hold for all $\eta \in U$ and $\rho \in H^2(\mathbb{R})$.

(iii) The quantities

$$\mathcal{K}_{\mathrm{r}}(\eta) = \sum_{j=4}^{\infty}\mathcal{K}_j(\eta), \qquad \mathcal{L}_{\mathrm{r}}(\eta) = \sum_{j=4}^{\infty}\mathcal{L}_j(\eta)$$

satisfy the estimates

$$\|\mathcal{K}_{\mathrm{r}}(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3\|\eta\|_2, \qquad \|\mathrm{d}\mathcal{K}_r[\eta](\rho)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3\|\rho\|_2 + \|\eta\|_{\mathcal{Z}}^2\|\eta\|_2\|\rho\|_2,$$
$$\|\mathcal{L}_{\mathrm{r}}(\eta)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3\|\eta\|_2, \qquad \|\mathrm{d}\mathcal{L}_r[\eta](\rho)\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^3\|\rho\|_2 + \|\eta\|_{\mathcal{Z}}^2\|\eta\|_2\|\rho\|_2$$

hold for all $\eta \in U$ and $\rho \in H^2(\mathbb{R})$.

Proof. These results are obtained by estimating the right-hand sides of (43), (44), (46), (47) and

$$\begin{aligned}
\mathcal{K}_{\mathrm{r}}(\eta) &= \mathcal{K}(\eta) - \mathcal{K}_1(\eta) - \mathcal{K}_2(\eta) - \mathcal{K}_3(\eta)\\
&= -\left(\gamma\nu\left(\frac{1}{1+\eta}\right) - \gamma\nu(1) + \gamma\eta + (A_0-1)\eta^2 + (B_0+1)\eta^3\right)\\
&\quad - \left(\frac{1}{(1+\eta_z^2)^{3/2}} - 1 + \tfrac{3}{2}\eta_z^2\right)\eta_{zz} + (1-\eta)\left(\frac{1}{(1+\eta_z^2)^{1/2}} - 1 + \tfrac{1}{2}\eta_z^2\right)\\
&\quad + (\eta^2-\eta^3)\left(\frac{1}{(1+\eta_z^2)^{1/2}} - 1\right) + \left(\frac{1}{1+\eta} - 1 + \eta - \eta^2 + \eta^3\right)\frac{1}{(1+\eta_z^2)^{1/2}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{r}}(\eta) &= \mathcal{L}(\eta) - \mathcal{L}_1(\eta) - \mathcal{L}_2(\eta) - \mathcal{L}_3(\eta)\\
&= -\tfrac{1}{2}(K_{\geq 2}(\eta)\eta + \tfrac{1}{2}K(\eta)\eta^2)^2 - (K_0\eta + K_1(\eta)\eta)(K_{\geq 2}(\eta)\eta + \tfrac{1}{2}K_{\geq 1}(\eta)\eta^2) - \tfrac{1}{2}\left(K_1(\eta)\eta\right)^2\\
&\quad - \tfrac{1}{2}K_1(\eta)\eta(K_0\eta^2) + \frac{1}{2}\left(\frac{1}{1+\eta_z^2} - 1\right)(\eta_z - \eta_z K(\eta)\eta - \tfrac{1}{2}\eta_z K(\eta)\eta^2)^2\\
&\quad + \tfrac{1}{2}(\eta_z K(\eta)\eta + \tfrac{1}{2}\eta_z K(\eta)\eta^2)^2 - \eta_z^2(K_{\geq 1}(\eta)\eta + \tfrac{1}{2}K(\eta)\eta^2)\\
&\quad + K_{\geq 3}(\eta)\eta + \tfrac{1}{2}K_{\geq 2}(\eta)\eta^2
\end{aligned}$$

with

$$K_{\geq 1}(\eta) = \sum_{j=1}^{\infty}K_j(\eta), \qquad K_{\geq 2}(\eta) = \sum_{j=2}^{\infty}K_j(\eta), \qquad K_{\geq 3}(\eta) = \sum_{j=3}^{\infty}K_j(\eta)$$

using the methods described in the proof of Corollary 2.5, noting that

$$\|K_j(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}^j\|\eta\|_{3/2}, \qquad \|K(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{3/2}$$

and

$$\|K_{\geq 1}(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}\|\eta\|_{3/2}, \qquad \|K_{\geq 2}(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_{3/2}, \qquad \|K_{\geq 3}(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}^3\|\eta\|_{3/2}.$$

The estimates for the derivatives are obtained in the same way. □

3 Reduction

In this section we reduce the equation

$$\mathcal{K}(\eta) - c_0^2(1-\varepsilon^2)\mathcal{L}(\eta) = 0 \tag{48}$$

to a perturbation of a full-dispersion model equation using a technique reminiscent of the Lyapunov-Schmidt reduction. We work in the subset U of the basic space $\mathcal{X} = H^2(\mathbb{R}^2)$ (see equation (40), so that equation (48) holds in $L^2(\mathbb{R}^2)$. Respecting the decomposition of η into two parts, we decompose $\mathcal{X}$ into the direct sum of the spaces

$$\mathcal{X}_1 = \chi(D)\mathcal{X}, \qquad \mathcal{X}_2 = (1-\chi(D))\mathcal{X}$$

and equip $\mathcal{X}_1$ and $\mathcal{X}_2$ with respectively the scaled norm

$$|||\eta_1|||^2 = \int_{\mathbb{R}} (1+\varepsilon^{-2}(|k|-\omega)^2)|\hat{\eta}_1|^2 \,\mathrm{d}k$$

(with the convention that $\omega = 0$ if $1 < \gamma < 9$) and the usual norm for $H^2(\mathbb{R})$.

Proposition 3.1 *The estimate*

$$\|\hat{\eta}\|_{L^1\mathbb{R})} \lesssim \varepsilon^{1/2}|||\eta_1|||$$

holds for every $\eta \in \mathcal{X}_1$, and in particular

$$\|\eta\|_{\mathcal{Z}} \lesssim \varepsilon^{1/2}|||\eta_1||| + \|\eta_2\|_2$$

for every $\eta \in \mathcal{X}$.

Proof. This result follows from the calculation

$$\begin{aligned}\int_{\mathbb{R}} |\hat{\eta}_1(k)| \,\mathrm{d}k &= \int_{\mathbb{R}} \frac{(1+\varepsilon^{-2}(|k|-\omega)^2)^{1/2}}{(1+\varepsilon^{-2}(|k|-\omega)^2)^{1/2}} |\hat{\eta}_1(k)| \,\mathrm{d}k \\ &\leq \left(\int_{\mathbb{R}} \frac{1}{1+\varepsilon^{-2}(|k|-\omega)^2} \,\mathrm{d}k\right)^{1/2} |||\eta_1||| \\ &= \left(\pi\varepsilon + 2\varepsilon \arctan \frac{\omega}{\varepsilon}\right)^{1/2} |||\eta_1|||.\end{aligned}$$

□

Clearly $\eta \in U$ satisfies (48) if and only if

$$\begin{aligned}\chi(D)\left(\mathcal{K}(\eta_1+\eta_2) - c_0^2(1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2)\right) &= 0, \\ (1-\chi(D))\left(\mathcal{K}(\eta_1+\eta_2) - c_0^2(1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2)\right) &= 0,\end{aligned}$$

and these equations can be rewritten as

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\mathcal{N}(\eta_1+\eta_2) = 0, \tag{49}$$

$$g(D)\eta_2 + c_0^2\varepsilon^2 K_0\eta_2 + (1-\chi(D))\mathcal{N}(\eta_1+\eta_2) = 0, \tag{50}$$

in which

$$\mathcal{N}(\eta) = \mathcal{K}_2(\eta) + \mathcal{K}_3(\eta) + \mathcal{K}_{\mathrm{r}}(\eta) - c_0^2(1-\varepsilon^2)(\mathcal{L}_2(\eta) + \mathcal{L}_3(\eta) + \mathcal{L}_{\mathrm{r}}(\eta)).$$

We proceed by writing (50) as a fixed-point equation for η_2 using Proposition 3.2, which follows from the fact that $g(k) \gtrsim |k|^2$ for $k \notin S$, and solving it for η_2 as a function of η_1 using Theorem 3.3, which is proved by a straightforward application of the contraction mapping principle. Substituting $\eta_2 = \eta_2(\eta_1)$ into (49) yields a reduced equation for η_1, which can be rewritten as a perturbation of a full-dispersion model equation by applying a further change of variable. Full details are given in Sections 3.1 and 3.2 below, which deal with the cases $1 < \gamma < 9$ ('strong surface tension') and $\gamma > 9$ ('weak surface tension') separately.

Proposition 3.2 *The mapping $(1-\chi(D))g(D)^{-1}$ is a bounded linear operator $L^2(\mathbb{R}) \to \mathcal{X}_2$.*

Theorem 3.3 *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be Banach spaces, X_1, X_2 be closed, convex sets in, respectively, $\mathcal{X}_1$, $\mathcal{X}_2$ containing the origin and $\mathcal{G}\colon X_1 \times X_2 \to \mathcal{X}_2$ be a smooth function. Suppose that there exists a continuous function $r\colon X_1 \to [0,\infty)$ such that*

$$\|\mathcal{G}(x_1, 0)\| \leq \tfrac{1}{2}r, \quad \|\mathrm{d}_2\mathcal{G}[x_1, x_2]\| \leq \tfrac{1}{3}$$

for each $x_2 \in \bar{B}_r(0) \subseteq X_2$ and each $x_1 \in X_1$.

Under these hypotheses there exists for each $x_1 \in X_1$ a unique solution $x_2 = x_2(x_1)$ of the fixed-point equation $x_2 = \mathcal{G}(x_1, x_2)$ satisfying $x_2(x_1) \in \bar{B}_r(0)$. Moreover $x_2(x_1)$ is a smooth function of $x_1 \in X_1$ and in particular satisfies the estimate

$$\|\mathrm{d}x_2[x_1]\| \leq 2\|\mathrm{d}_1\mathcal{G}[x_1, x_2(x_1)]\|.$$

3.1 Strong surface tension

Suppose that $1 < \gamma < 9$. We write (50) in the form

$$\eta_2 = -(1 - \chi(D))g(D)^{-1}\left(c_0^2\varepsilon^2 K_0\eta_2 + \mathcal{N}(\eta_1 + \eta_2)\right) \tag{51}$$

and apply Theorem 3.3 with

$$X_1 = \{\eta_1 \in \mathcal{X}_1 \colon |||\eta_1||| \leq R_1\}, \qquad X_2 = \{\eta_2 \in \mathcal{X}_2 \colon \|\eta_2\|_2 \leq R_2\};$$

the function $\mathcal{G}$ is given by the right-hand side of (51). Using Proposition 3.1 one can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < \frac{1}{2}M$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_2 is constrained by the requirement that $\|\eta_2\|_2 < \frac{1}{2}M$ for all $\eta_2 \in X_2$. The next lemma follows from Lemma 2.17, its corollary from Proposition 3.2.

Lemma 3.4 *The estimates*

(i) $\|\mathcal{N}(\eta_1, \eta_2)\|_0 \lesssim \varepsilon^{1/2}|||\eta_1|||^2 + \varepsilon^{1/2}|||\eta_1|||\,\|\eta_2\|_2 + |||\eta_1|||\,\|\eta_2\|_2^2 + \|\eta_2\|_2^2$,

(ii) $\|\mathrm{d}_1\mathcal{N}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2}|||\eta_1||| + \varepsilon^{1/2}\|\eta_2\|_2 + \|\eta_2\|_2^2$,

(iii) $\|\mathrm{d}_2\mathcal{N}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_2, L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2}|||\eta_1||| + |||\eta_1|||\,\|\eta_2\|_2 + \|\eta_2\|_2$,

where with a slight abuse of notation we write $\mathcal{N}(\eta_1 + \eta_2)$ as $\mathcal{N}(\eta_1, \eta_2)$, hold for each $\eta_1 \in X_1$ and $\eta_2 \in X_2$.

Corollary 3.5 *The estimates*

(i) $\|\mathcal{G}(\eta_1, \eta_2)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2 + \varepsilon^{1/2}|||\eta_1|||\,\|\eta_2\|_2 + |||\eta_1|||\,\|\eta_2\|_2^2 + \|\eta_2\|_2^2 + \varepsilon^2\|\eta_2\|_2$,

(ii) $\|\mathrm{d}_1\mathcal{G}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1||| + \varepsilon^{1/2}\|\eta_2\|_2 + \|\eta_2\|_2^2$,

(iii) $\|\mathrm{d}_2\mathcal{G}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1||| + |||\eta_1|||\,\|\eta_2\|_2 + \|\eta_2\|_2 + \varepsilon^2$

hold for each $\eta_1 \in X_1$ and $\eta_2 \in X_2$.

Theorem 3.6 *Equation* (51) *has a unique solution $\eta_2 \in X_2$ which depends smoothly upon $\eta_1 \in X_1$ and satisfies the estimates*

$$\|\eta_2(\eta_1)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2, \qquad \|\mathrm{d}\eta_2[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1|||.$$

Proof. Choosing R_2 and ε sufficiently small and setting $r(\eta_1) = \sigma\varepsilon^{1/2}|||\eta_1|||^2$ for a sufficiently large value of $\sigma > 0$, one finds that

$$\|\mathcal{G}(\eta_1, 0)\|_2 \lesssim \tfrac{1}{2}r(\eta_1), \qquad \|\mathrm{d}_2\mathcal{G}[\eta_1, \eta_3]\|_{\mathcal{L}(\mathcal{X}_2, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}$$

for $\eta_1 \in X_1$ and $\eta_2 \in \overline{B}_{r(\eta_1)}(0) \subset X_2$ (Corollary 3.5(i), (iii)). Theorem 3.3 asserts that equation (51) has a unique solution η_2 in $\overline{B}_{r(\eta_1)}(0) \subset X_2$ which depends smoothly upon $\eta_1 \in X_1$, and the estimate for its derivative follows from Corollary 3.5(ii). $\square$

Substituting $\eta_2 = \eta_2(\eta_1)$ into (49) yields the reduced equation

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\mathcal{N}(\eta_1 + \eta_2(\eta_1)) = 0 \tag{52}$$

for $\eta_1 \in X_1$. The leading-order terms in this equation are computed by approximating the operators ∂_z and K_0 in its quadratic part by constants.

Proposition 3.7 *The estimates*

(i) $\eta_{1z} = O(\varepsilon|||\eta_1|||)$,

(ii) $K_0\eta_1 = 2\eta_1 + O(\varepsilon|||\eta_1|||)$,

(iii) $K_0(\eta_1\rho_1) = 2\eta_1\rho_1 + O(\varepsilon^{3/2}|||\eta_1|||\,|||\rho_1|||)$

hold for all η_1, $\rho_1 \in \mathcal{X}_1$. The order-of-magnitude estimates are computed with respect to the $L^2(\mathbb{R})$-norm (which is equivalent to the $H^s(\mathbb{R})$-norm on the space $\chi(D)H^s(\mathbb{R})$ for any $s \geq 0$).

Proof. This result follows from the calculations

$$\begin{aligned}
\|\eta_{1z}\|_0 &= \||k|\hat{\eta}_1\|_0 \leq \varepsilon|\!|\!|\eta_1|\!|\!|,\\
\|(K_0-2I)\eta_1\|_0 &= \|(f(k)-2)\hat{\eta}_1\|_0 \lesssim \||k|\hat{\eta}_1\|_0 \leq \varepsilon|\!|\!|\eta_1|\!|\!|,\\
\|(K_0-2I)(\eta_1\rho_1)\|_0 &\lesssim \left\||k|\int_{\mathbb{R}} |\hat{\eta}_1(k-\tilde{k})||\hat{\rho}_1(\tilde{k})|\,\mathrm{d}\tilde{k}\right\|_0\\
&\lesssim \left\|\int_{\mathbb{R}} |k-\tilde{k}||\hat{\eta}_1(k-\tilde{k})||\hat{\rho}_1(\tilde{k})|\,\mathrm{d}\tilde{k} + \int_{\mathbb{R}} |\tilde{k}||\hat{\eta}_1(k-\tilde{k})||\hat{\rho}_1(\tilde{k})|\,\mathrm{d}\tilde{k}\right\|_0\\
&\lesssim \|\,|k|\hat{\eta}_1\|_0\|\hat{\rho}_1\|_{L^1(\mathbb{R})} + \|\hat{\eta}_1\|_{L^1(\mathbb{R})}\|\,|k|\hat{\rho}_1\|_0\\
&\lesssim \varepsilon^{3/2}|\!|\!|\eta_1|\!|\!|\,|\!|\!|\rho_1|\!|\!|
\end{aligned}$$

for each η_1, $\rho_1 \in X_1$, where we have also used Young's inequality. □

The leading-order terms in the nonlinear part of (52) are now obtained from Corollary 3.8 (which follows from Corollary 2.16 and Proposition 3.7) and Lemma 3.9 (which follows from Lemma 2.17) below. Here we use the symbol $\underline{O}(\varepsilon^s|\!|\!|\eta_1|\!|\!|^t)$ (with $s \geq 0$, $t \geq 1$) to denote a smooth function $\mathcal{R}^\varepsilon : X_1 \to L^2(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}^\varepsilon(\eta_1)\|_0 \lesssim \varepsilon^s|\!|\!|\eta_1|\!|\!|^t \quad \|\mathrm{d}\mathcal{R}^\varepsilon[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,L^2(\mathbb{R}^2))} \lesssim \varepsilon^s|\!|\!|\eta_1|\!|\!|^{t-1}$$

for each $\eta_1 \in X_1$.

Corollary 3.8 *The estimates*

(i) $\mathcal{K}_2(\eta_1+\eta_2(\eta_1)) = \left(-\gamma - \tfrac{1}{2}\gamma v''(1) + 1\right)\eta_1^2 + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|^2)$,

(ii) $\mathcal{L}_2(\eta_1+\eta_2(\eta_1)) = -5\eta_1^2 + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|^2)$

hold for each $\eta_1 \in X_1$.

Lemma 3.9 *The estimate*

$$\mathcal{N}(\eta_1+\eta_2(\eta_1)) = \mathcal{K}_2(\eta_1) - c_0^2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|^3)$$

holds for each $\eta_1 \in X_1$.

We conclude that the reduced equation for η_1 is the *perturbed full dispersion Korteweg-de Vries equation*

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\Big(2c_0^2 d_0\eta_1^2 + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|^2)\Big) = 0,$$

and applying Proposition 3.7(ii), one can further simplify it to

$$g(D)\eta_1 + 2c_0^2\varepsilon^2\eta_1 + \chi(D)\Big(2c_0^2 d_0\eta_1^2 + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|^2) + \underline{O}(\varepsilon^3|\!|\!|\eta_1|\!|\!|)\Big) = 0.$$

Finally, we introduce the Korteweg-de Vries scaling

$$\eta_1(z) = \varepsilon^2\zeta(\varepsilon z),$$

noting that $I : \eta_1 \to \zeta$ is an isomorphism $\mathcal{X}_1 \to H^1_\varepsilon(\mathbb{R})$ and $\chi(D)L^2(\mathbb{R}) \to L^2_\varepsilon(\mathbb{R})$ and choosing $R > 1$ large enough so that $\zeta_{\mathrm{KdV}} \in B_R(0)$ (and $\varepsilon > 0$ small enough so that $B_R(0) \subseteq H^1_\varepsilon(\mathbb{R})$ is contained in $I[X_1]$). We find that $\zeta \in B_R(0) \subseteq H^1_\varepsilon(\mathbb{R})$ satisfies the equation

$$\varepsilon^{-2}g(\varepsilon D)\zeta + 2c_0^2\zeta + 2c_0^2 d_0\chi_0(\varepsilon D)\zeta^2 + \varepsilon^{1/2}\underline{O}_0^\varepsilon(\|\zeta\|_1) = 0, \tag{53}$$

which holds in $L^2_\varepsilon(\mathbb{R})$, where the symbol D now means $-\mathrm{i}\partial_Z$ and the symbol $\underline{O}_n^\varepsilon(\varepsilon^s\|\zeta\|_1^t)$ denotes a smooth function $\mathcal{R} : B_R(0) \subseteq H^1_\varepsilon(\mathbb{R}) \to H^n_\varepsilon(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}(\zeta)\|_n \lesssim \varepsilon^s\|\zeta\|_1^t \quad \|\mathrm{d}\mathcal{R}[\zeta]\|_{\mathcal{L}(H^1(\mathbb{R}),H^n(\mathbb{R}))} \lesssim \varepsilon^s\|\zeta\|_1^{t-1}$$

for each $\zeta \in B_R(0) \subseteq H^1_\varepsilon(\mathbb{R})$ (with $t \geq 1$, $s, n \geq 0$). Note that $|\!|\!|\eta|\!|\!| = \varepsilon^{3/2}\|\zeta\|_1$ and the change of variable from z to $Z = \varepsilon z$ introduces an additional factor of $\varepsilon^{1/2}$ in the remainder term.

Equation (52) is invariant under the reflection $\eta_1(z) \mapsto \eta_1(-z)$; a familiar argument shows that it is inherited from the corresponding invariance of (49), (51) under $\eta_1(z) \mapsto \eta_1(-z)$, $\eta_2(z) \mapsto \eta_2(-z)$ when applying Theorem 3.3. The invariance is likewise inherited by (53), which is invariant under the reflection $\zeta(Z) \mapsto \zeta(-Z)$.

3.2 *Weak surface tension*

Suppose that $\gamma > 9$. Since $\chi(D)\mathcal{K}_2(\eta_1)$ and $\chi(D)\mathcal{L}_2(\eta_1)$ both vanish the nonlinear term in (49) is at leading order cubic in η_1, so that this equation may be rewritten as

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\left(\mathcal{N}(\eta_1+\eta_2) + c_0^2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) - \mathcal{K}_2(\eta_1)\right) = 0. \tag{54}$$

To compute the reduced equation for η_1 we need an explicit formula for the leading-order quadratic part of $\eta_2(\eta_1)$; inspecting (50) shows that it is given by

$$F(\eta_1) := (1-\chi(D))g(D)^{-1}\left(c_0^2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) - \mathcal{K}_2(\eta_1)\right), \tag{55}$$

an estimate for which is found using Lemma 2.17 (note that $K_0F(\eta)$ satisfies the same estimates as $F(\eta)$ since $\mathcal{F}[F(\eta)]$ has compact support).

Proposition 3.10 *The estimates*

(i) $\|F(\eta_1)\|_2, \|K_0F(\eta_1)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2$,

(ii) $\|\mathrm{d}F[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)}, \|\mathrm{d}K_0F[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1|||$

hold for each $\eta_1 \in X_1$.

It is convenient to write $\eta_2 = F(\eta_1)+\eta_3$ and (50) in the form

$$\eta_3 = -(1-\chi(D))g(D)^{-1}\Big(\mathcal{N}(\eta_1+F(\eta_1)+\eta_3) + c_0^2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) - \mathcal{K}_2(\eta_1) + c_0^2\varepsilon^2K_0(F(\eta_1)+\eta_3)\Big) \tag{56}$$

(with the requirement that $\eta_1 + F(\eta_1)+\eta_3 \in U$). We apply Theorem 3.3 to equation (56) with

$$X_1 = \{\eta_1 \in \mathcal{X}_1 \colon |||\eta_1||| \leq R_1\}, \qquad X_3 = \{\eta_3 \in \mathcal{X}_2 \colon \|\eta_3\|_3 \leq R_3\};$$

the function $\mathcal{G}$ is given by the right-hand side of (56). (Here we write X_3 rather than X_2 for notational clarity.) Using Proposition 3.1 one can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R})} < \frac{1}{2}M$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_3 is constrained by the requirement that $\|F(\eta_1)+\eta_3\|_2 < \frac{1}{2}M$ for all $\eta_1 \in X_1$ and $\eta_3 \in X_3$, so that $\eta_1 + F(\eta_1) + \eta_3 \in U$ (Proposition 3.10 asserts that $\|F(\eta_1)\|_2 = O(\varepsilon^{1/2})$ uniformly over $\eta_1 \in X_1$). We proceed by writing

$$\mathcal{N}(\eta_1+F(\eta_1)+\eta_3) + c_0^2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) - \mathcal{K}_2(\eta_1) = -c_0^2(1-\varepsilon^2)\mathcal{N}_1(\eta_1,\eta_3) + \mathcal{N}_2(\eta_1,\eta_3) + \mathcal{N}_3(\eta_1,\eta_3),$$

where

$$\begin{aligned}
\mathcal{N}_1(\eta_1,\eta_3) &= \mathcal{L}_2(\eta_1+F(\eta_1)+\eta_3) - \mathcal{L}_2(\eta_1),\\
\mathcal{N}_2(\eta_1,\eta_3) &= \mathcal{K}_2(\eta_1+F(\eta_1)+\eta_3) - \mathcal{K}_2(\eta_1),\\
\mathcal{N}_3(\eta_1,\eta_3) &= \mathcal{K}_3(\eta_1+F(\eta_1)+\eta_3) + \mathcal{K}_\mathrm{r}(\eta_1+F(\eta_1)+\eta_3)\\
&\qquad - c_0^2(1-\varepsilon^2)\left(\mathcal{L}_3(\eta_1+F(\eta_1)+\eta_3) + \mathcal{L}_\mathrm{r}(\eta_1+F(\eta_1)+\eta_3)\right)
\end{aligned}$$

and estimating these quantities using Lemma 2.17.

Proposition 3.11 *The estimates*

(i) $\|\mathcal{N}_1(\eta_1,\eta_3)\|_0, \|\mathcal{N}_2(\eta_1,\eta_3)\|_0 \lesssim \varepsilon|||\eta_1|||^3 + \varepsilon^{1/2}|||\eta_1|||^2\|\eta_3\|_2 + \varepsilon^{1/2}|||\eta_1|||\|\eta_3\|_2 + \|\eta_3\|_2^2$,

(ii) $\|\mathrm{d}_1\mathcal{N}_1[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,L^2(\mathbb{R}))}, \|\mathrm{d}_1\mathcal{N}_2[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,L^2(\mathbb{R}))} \lesssim \varepsilon|||\eta_1|||^2 + \varepsilon^{1/2}|||\eta_1|||\|\eta_3\|_2 + \varepsilon^{1/2}\|\eta_3\|_2$,

(iii) $\|\mathrm{d}_2\mathcal{N}_1[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,L^2(\mathbb{R}))}, \|\mathrm{d}_2\mathcal{N}_2[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_2$

and

(iv) $\|\mathcal{N}_3(\eta_1,\eta_3)\|_0 \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_3)^2(|||\eta_1||| + \|\eta_3\|_3)$,

(v) $\|\mathrm{d}_1\mathcal{N}_3[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,L^2(\mathbb{R}))} \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_3)^2$,

(vi) $\|\mathrm{d}_2\mathcal{N}_3[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_3,L^2(\mathbb{R}))} \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_3)(|||\eta_1||| + \|\eta_3\|_3)$

hold for each $\eta_1 \in X_1$ *and* $\eta_3 \in X_3$.

The final estimates for $\mathcal{G}$ and its derivatives follow from Propositions 3.10 and 3.11 by virtue of Proposition 3.2.

Corollary 3.12 *The estimates*

(i) $\|\mathcal{G}(\eta_1,\eta_3)\|_2 \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_2)^2(1+|||\eta_1|||+\|\eta_3\|_2) + \varepsilon^2\|\eta_3\|_2$,

(ii) $\|\mathrm{d}_1\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_2)(\varepsilon^{1/2} + \varepsilon^{1/2}|||\eta_1|||+\|\eta_3\|_2)$,

(iii) $\|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_2)} \lesssim (\varepsilon^{1/2}|||\eta_1||| + \|\eta_3\|_2)(1+|||\eta_1|||+\|\eta_3\|_2) + \varepsilon^2$

hold for each $\eta_1 \in X_1$ *and* $\eta_3 \in X_3$.

Theorem 3.13 *Equation* (56) *has a unique solution* $\eta_3 \in X_3$ *which depends smoothly upon* $\eta_1 \in X_1$ *and satisfies the estimates*

$$\|\eta_3(\eta_1)\|_2 \lesssim \varepsilon|||\eta_1|||^2, \qquad \|\mathrm{d}\eta_3[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon|||\eta_1|||.$$

Proof. Choosing R_3 and ε sufficiently small and setting $r(\eta_1) = \sigma\varepsilon|||\eta_1|||^2$ for a sufficiently large value of $\sigma>0$, one finds that

$$\|\mathcal{G}(\eta_1,0)\|_2 \lesssim \tfrac{1}{2}r(\eta_1), \qquad \|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_2)} \lesssim \varepsilon^{1/2}$$

for $\eta_1 \in X_1$ and $\eta_3 \in \overline{B}_{r(\eta_1)}(0) \subset X_3$ (Lemma 3.12(i), (iii)). Theorem 3.3 asserts that equation (56) has a unique solution η_3 in $\overline{B}_{r(\eta_1)}(0) \subset X_3$ which depends smoothly upon $\eta_1 \in X_1$, and the estimate for its derivative follows from Lemma 3.12(ii). □

Substituting $\eta_2 = F(\eta_1) + \eta_3(\eta_1)$ into (54) yields the reduced equation

$$g(D)\eta_1 + c_0^2\varepsilon^2 K_0\eta_1 + \chi(D)\left(-c_0^2(1-\varepsilon^2)\mathcal{N}_1(\eta_1,\eta_3(\eta_1)) + \mathcal{N}_2(\eta_1,\eta_3(\eta_1)) + \mathcal{N}_3(\eta_1,\eta_3(\eta_1))\right) = 0 \qquad (57)$$

for $\eta_1 \in X_1$. The next step is to compute the leading-order terms in the reduced equation. To this end we write

$$\eta_1 = \eta_1^+ + \eta_1^-,$$

where $\eta_1^\pm = \chi^\pm(D)\eta_1$ and $\chi^\pm(D)$ are the characteristic functions of the sets $(\pm\omega-\delta, \pm\omega+\delta)$, so that η_1^+ satisfies the equation

$$g(D)\eta_1^+ + c_0^2\varepsilon^2 K_0\eta_1^+ + \chi^+(D)\left(-c_0^2(1-\varepsilon^2)\mathcal{N}_1(\eta_1,\eta_3(\eta_1)) + \mathcal{N}_2(\eta_1,\eta_3(\eta_1)) + \mathcal{N}_3(\eta_1,\eta_3(\eta_1))\right) = 0 \qquad (58)$$

(and $\eta_1^- = \overline{\eta_1^+}$ satisfies its complex conjugate). We again begin by showing how Fourier-multiplier operators acting upon the function η_1 may be approximated by constants. The following result is proved in the same way as Proposition 3.7.

Proposition 3.14 *The estimates*

(i) $\partial_z\eta_1^\pm = \pm\mathrm{i}\omega\eta_1^\pm + O(\varepsilon|||\eta_1|||)$,

(ii) $\partial_z^2\eta_1^\pm = -\omega^2\eta_1^\pm + O(\varepsilon|||\eta_1|||)$,

(iii) $K_0\eta_1^\pm = f(\omega)\eta_1^\pm + O(\varepsilon|||\eta_1|||)$,

(iv) $K_0(\eta_1^+\rho_1^+) = f(2\omega)(\eta_1^+\rho_1^+) + O(\varepsilon^{3/2}|||\eta_1||| \, |||\rho_1|||)$,

(v) $K_0(\eta_1^+\rho_1^-) = 2\eta_1^+\rho_1^- + O(\varepsilon^{3/2}|||\eta_1||| \, |||\rho_1|||)$,

(vi) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[\eta_1^+\rho_1^+]] = g(2\omega)^{-1}(\eta_1^+\rho_1^+) + O(\varepsilon^{3/2}|||\eta_1||| \, |||\rho_1|||)$,

(vii) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[\eta_1^+\rho_1^-]] = g(0)^{-1}\eta_1^+\rho_1^- + O(\varepsilon^{3/2}|||\eta_1||| \, |||\rho_1|||)$,

(viii) $K_0(\eta_1^+\rho_1^+\xi_1^-) = f(\omega)(\eta_1^+\rho_1^+\xi_1^-) + O(\varepsilon^2|||\eta_1||| \, |||\rho_1||| \, |||\xi_1|||)$

hold for all η_1, ρ_1, $\xi_1 \in \mathcal{X}_1$, *where the order-of-magnitude estimates are computed with respect to the* $L^2(\mathbb{R})$*-norm.*

We proceed by approximating each term in the quadratic and cubic parts of equation (58) using Corollary 2.16 and Lemma 3.14.

Proposition 3.15 *The estimate*

$$F(\eta_1) = g(2\omega)^{-1}\left(c_0^2 A(\omega) - A_0 - \tfrac{1}{2}\omega^2\right)\left((\eta_1^+)^2 + (\eta_1^-)^2\right) + g(0)^{-1}\left(c_0^2 B(\omega) - 2A_0 + \omega^2\right)\eta_1^+\eta_1^- + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^2),$$

where

$$A(\omega) = \tfrac{3}{2}\omega^2 - \tfrac{1}{2}f(\omega)^2 - f(\omega)f(2\omega) + \tfrac{1}{2}f(2\omega), \qquad B(\omega) = \omega^2 - f(\omega)^2 - 4f(\omega) + 2,$$

holds for each $\eta_1 \in \mathcal{X}_1$.

Proposition 3.16 *The estimate*

$$\begin{aligned}
&\chi^+(D)\left(c_0^2(1-\varepsilon^2)\mathcal{N}_1(\eta_1,\eta_3) - \mathcal{N}_2(\eta_1,\eta_3)\right) \\
&\quad = \chi^+(D)\Big(\big(2g(2\omega)^{-1}(c_0^2 C(\omega) - A_0 + \omega^2)(c_0^2 A(\omega) - A_0 - \tfrac{1}{2}\omega^2) \\
&\qquad\qquad + 2g(0)^{-1}(c_0^2 D(\omega) - A_0)(c_0^2 B(\omega) - 2A_0 + \omega^2)\big)(\eta_1^+)^2\eta_1^- + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3)\Big),
\end{aligned}$$

where

$$C(\omega) = \tfrac{3}{2}\omega^2 - f(\omega)f(2\omega) + \tfrac{1}{2}f(\omega) - \tfrac{1}{2}f(\omega)^2, \qquad D(\omega) = \tfrac{1}{2}\omega^2 - \tfrac{3}{2}f(\omega) - \tfrac{1}{2}f(\omega^2),$$

holds for each $\eta_1 \in X_1$.

Proposition 3.17 *The estimates*

(i) $\chi^+(D)\mathcal{K}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \chi^+(D)\Big(\left(3B_0 + \tfrac{1}{2}\omega^2 - \tfrac{3}{2}\omega^4\right)(\eta_1^+)^2\eta_1^- + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3)\Big)$,

(ii) $\chi^+(D)\mathcal{L}_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \chi^+(D)\Big(E(\omega)(\eta_1^+)^2\eta_1^- + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3)\Big)$,

where

$$E(\omega) = 2f(\omega)^2 f(2\omega) - 6f(\omega)\omega^2 + \tfrac{13}{2}f(\omega)^2 - f(\omega)f(2\omega) - 4f(\omega) + \tfrac{1}{2}\omega^2,$$

hold for each $\eta_1 \in X_1$.

The higher-order terms in equation are estimated using Lemma 2.17(iii).

Proposition 3.18 *The estimates*

(i) $\mathcal{K}_{\mathrm{r}}(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \underline{O}(\varepsilon^2|||\eta_1|||^4)$,

(ii) $\mathcal{L}_{\mathrm{r}}(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \underline{O}(\varepsilon^{3/2}|||\eta_1|||^4)$

hold for each $\eta_1 \in X_1$.

Corollary 3.19 *The estimate*

$$\chi^+(D)\mathcal{N}_3(\eta_1,\eta_3(\eta_1)) = \chi^+(D)\Big(\left(3B_0 + \tfrac{1}{2}\omega^2 - \tfrac{3}{2}\omega^4 - c_0^2 E(\omega)\right)(\eta_1^+)^2\eta_1^- + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3)\Big)$$

holds for each $\eta_1 \in X_1$.

We conclude that the reduced equation for η_1 is the *perturbed full dispersion nonlinear Schrödinger equation*

$$g(D)\eta_1^+ + c_0^2\varepsilon^2 K_0\eta_1^+ + \chi^+(D)\Big(-4a_3|\eta_1^+|^2\eta_1^+ + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3)\Big) = 0,$$

where

$$\begin{aligned}
4a_3 &= 2g(2\omega)^{-1}(c_0^2 C(\omega) - A_0 + \omega^2)(c_0^2 A(\omega) - A_0 - \tfrac{1}{2}\omega^2) \\
&\quad + 2g(0)^{-1}(c_0^2 D(\omega) - A_0)(c_0^2 B(\omega) - 2A_0 + \omega^2) - 3B_0 - \tfrac{1}{2}\omega^2 + \tfrac{3}{2}\omega^4 + c_0^2 E(\omega),
\end{aligned}$$

and applying Lemma 3.14(iii), one can further simplify it to

$$g(D)\eta_1^+ + c_0^2 f(\omega)\varepsilon^2\eta_1^+ + \chi^+(D)\Big(-4a_3|\eta_1^+|^2\eta_1^+ + \underline{O}(\varepsilon^{3/2}|||\eta_1|||^3) + \underline{O}(\varepsilon^3|||\eta_1|||)\Big) = 0.$$

Finally, we introduce the nonlinear Schrödinger scaling

$$\eta_1^+(z) = \tfrac{1}{2}\varepsilon\zeta(\varepsilon z)\mathrm{e}^{\mathrm{i}\omega z},$$

noting that $I : \eta_1^+ \mapsto \zeta$ is an isomorphism $\mathcal{X}_1^+ := \chi^+(D)\mathcal{X}_1 \to H_\varepsilon^1(\mathbb{R})$ and $\chi^+(D)L^2(\mathbb{R}) \to L_\varepsilon^2(\mathbb{R})$, where $\mathcal{X}_1^+ = \chi(D)\mathcal{X}_1$, and choosing $R > 1$ large enough so that $\zeta_{\mathrm{NLS}} \in B_R(0)$ (and $\varepsilon > 0$ small enough so that $B_R(0) \subset H_\varepsilon^1(\mathbb{R})$ is contained in $I[\mathcal{X}_1^+]$). We find that $\zeta \in B_R(0) \subseteq H_\varepsilon^1(\mathbb{R})$ satisfies the equation

$$\varepsilon^{-2}g(\omega+\varepsilon D)\zeta + c_0^2 f(\omega)\zeta - a_3\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{O}_0^\varepsilon(\|\zeta\|_1) = 0, \tag{59}$$

which holds in $L_\varepsilon^2(\mathbb{R})$. Note that $|||\eta_1||| = \varepsilon^{1/2}\|\zeta\|_1$ and the change of variable from z to $Z = \varepsilon z$ introduces an additional factor of $\varepsilon^{1/2}$ in the remainder term. Equation (57) is of course also invariant under the reflection $\eta_1(z) \mapsto \eta_1(-z)$, and this invariance is inherited by (59), which is invariant under the reflection $\zeta(Z) \mapsto \overline{\zeta(-Z)}$.

4 Solution of the reduced equation

In this section we find solitary-wave solutions of the reduced equations

$$\varepsilon^{-2}g(\varepsilon D)\zeta + 2c_0^2\zeta + 2c_0^2 d_0\chi_0(\varepsilon D)\zeta^2 + \varepsilon^{1/2}\underline{O}_0^\varepsilon(\|\zeta\|_1) = 0, \tag{60}$$

and

$$\varepsilon^{-2}g(\omega+\varepsilon D)\zeta + c_0^2 f(\omega)\zeta - a_3\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{O}_0^\varepsilon(\|\zeta\|_1) = 0. \tag{61}$$

noting that in the formal limit $\varepsilon \to 0$ they reduce to respectively the stationary Korteweg-de Vries equation

$$(\tfrac{1}{8}\gamma - \tfrac{9}{8})\zeta_{ZZ} + 2c_0^2\zeta + 2c_0^2 d_0\zeta^2 = 0, \tag{62}$$

and the stationary nonlinear Schrödinger equation

$$-a_1\zeta_{ZZ} + a_2\zeta - a_3|\zeta|^2\zeta = 0, \tag{63}$$

which have explicit (symmetric) solitary-wave solutions ζ_{KdV} and $\pm\zeta_{\mathrm{NLS}}$ (equations (19) and (21)). For this purpose we use a perturbation argument, rewriting (60) and (61) as fixed-point equations and applying the following version of the implicit-function theorem. We again treat the cases $1 < \gamma < 9$ ('strong surface tension') and $\gamma > 9$ ('weak surface tension') separately.

Theorem 4.1 *Let $\mathcal{W}$ be a Banach space, W_0 and Λ_0 be open neighbourhoods of respectively $w^\star$ in $\mathcal{W}$ and the origin in $\mathbb{R}$ and $\mathcal{H} : W_0 \times \Lambda_0 \to \mathcal{W}$ be a function which is differentiable with respect to $w \in W_0$ for each $\lambda \in \Lambda_0$. Furthermore, suppose that $\mathcal{H}(w^\star, 0) = 0$, $\mathrm{d}_1\mathcal{H}[w^\star, 0] : \mathcal{W} \to \mathcal{W}$ is an isomorphism,*

$$\lim_{w \to w^\star} \|\mathrm{d}_1\mathcal{H}[w,0] - \mathrm{d}_1\mathcal{H}[w^\star,0]\|_{\mathcal{L}(\mathcal{W})} = 0$$

and

$$\lim_{\lambda\to 0}\|\mathcal{H}(w,\lambda) - \mathcal{H}(w,0)\|_{\mathcal{W}} = 0, \quad \lim_{\lambda\to 0}\|\mathrm{d}_1\mathcal{H}[w,\lambda] - \mathrm{d}_1\mathcal{H}[w,0]\|_{\mathcal{L}(\mathcal{W})} = 0$$

uniformly over $w \in X_0$.

There exist open neighbourhoods W of $w^\star$ in $\mathcal{W}$ and Λ of 0 in $\mathbb{R}$ (with $W \subseteq W_0$, $\Lambda \subseteq \Lambda_0$) and a uniquely determined mapping $h : \Lambda \to X$ with the properties that

(i) h is continuous at the origin (with $h(0) = w^\star$),

(ii) $\mathcal{H}(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,

(iii) $w = h(\lambda)$ whenever $(w, \lambda) \in W \times \Lambda$ satisfies $\mathcal{H}(w, \lambda) = 0$.

4.1 Strong surface tension

Theorem 4.2 *For each sufficiently small value of $\varepsilon > 0$ equation* (60) *has a small-amplitude, symmetric solution ζ_ε in $H^1_\varepsilon(\mathbb{R})$ with $\|\zeta_\varepsilon - \zeta_{\mathrm{KdV}}\|_1 \to 0$ as $\varepsilon \to 0$.*

The first step in the proof of Theorem 4.2 is to write (60) as the fixed-point equation

$$\zeta + \varepsilon^2\big(2c_0^2\varepsilon^2 + g(\varepsilon D)\big)^{-1}\Big(2c_0^2 d_0\chi_0(\varepsilon D)\zeta^2 + \varepsilon^{1/2}\underline{O}_0^\varepsilon(\|\zeta\|_1)\Big) = 0 \tag{64}$$

for $\zeta \in H^1_\varepsilon(\mathbb{R})$ and use the following elementary inequality to 'replace' the nonlocal operator with a differential operator.

Proposition 4.3 *The inequality*

$$\left|\frac{\varepsilon^2}{2c_0^2\varepsilon^2 + g(\varepsilon k)} - \frac{1}{2c_0^2 + (\frac{9}{8} - \frac{1}{9})k^2}\right| \lesssim \frac{\varepsilon}{(1+k^2)^{1/2}}$$

holds uniformly over $|k| < \delta/\varepsilon$.

Using the above proposition, one can write equation (64) as

$$\zeta + F_\varepsilon(\zeta) = 0,$$

where

$$F_\varepsilon(\zeta) = 2c_0^2 d_0\left(2c_0^2 - (\tfrac{9}{8} - \tfrac{1}{8}\gamma)\partial_Z^2\right)^{-1}\chi_0(\varepsilon D)\zeta^2 + \varepsilon^{1/2}\underline{O}_1^\varepsilon(\|\zeta\|_1).$$

It is convenient to replace this equation with

$$\zeta + \tilde{F}_\varepsilon(\zeta) = 0,$$

where $\tilde{F}_\varepsilon(\zeta) = F_\varepsilon(\chi_0(\varepsilon D)\zeta)$ and study it in the fixed space $H^1(\mathbb{R})$ (the solution sets of the two equations evidently coincide). We establish Theorem 4.6 by applying Theorem 4.1 with

$$\mathcal{W} = H^1_{\mathrm{e}}(\mathbb{R}) := \{u \in H^1(\mathbb{R}) : u(Z) = u(-Z) \text{ for all } Z \in \mathbb{R}\},$$

$W_0 = B_R(0)$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0, and

$$\mathcal{H}(\zeta, \varepsilon) := \zeta + \tilde{F}_{|\varepsilon|}(\zeta)$$

(here ε is replaced by $|\varepsilon|$ so that $\mathcal{H}(\zeta,\varepsilon)$ is defined for ε in a full neighbourhood of the origin in $\mathbb{R}$). Observe that

$$\mathcal{H}(\zeta,\varepsilon) - \mathcal{H}(\zeta,0) = 2c_0^2 d_0\left(2c_0^2 - (\tfrac{9}{8} - \tfrac{1}{8}\gamma)\partial_Z^2\right)^{-1}[\chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D)\zeta)^2 - \zeta^2] + |\varepsilon|^{1/2}\underline{O}_1^{|\varepsilon|}(\|\zeta\|_1),$$

and noting that

$$\lim_{\varepsilon\to 0}\|\chi_0(|\varepsilon|D) - I\|_{\mathcal{L}(H^1(\mathbb{R}), H^{3/4}(\mathbb{R}))} = 0$$

because

$$\begin{aligned}
\|\chi_0(|\varepsilon|D)u - u\|_{3/4}^2 &= \int_{|k| > \frac{\delta}{|\varepsilon|}} (1+|k|^2)^{3/4}|\hat{u}|^2\,\mathrm{d}k \\
&\leq \sup_{|k|>\frac{\delta}{|\varepsilon|}} (1+|k|^2)^{-1/4}\int_{|k|>\frac{\delta}{|\varepsilon|}} (1+|k|^2)|\hat{u}|^2\,\mathrm{d}k \\
&\leq \left(1 + \frac{\delta^2}{|\varepsilon|^2}\right)^{-1/4}\|u\|_1^2,
\end{aligned}$$

that

$$\chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D)\zeta)^2 - \zeta^2 = \chi_0(|\varepsilon|D)(\chi_0(|\varepsilon|D) + I)\zeta(\chi_0(|\varepsilon|D) - I)\zeta + (\chi_0(|\varepsilon|D) - I)\zeta^2$$

and that $H^{3/4}(\mathbb{R})$ is a Banach algebra, we find that

$$\lim_{\varepsilon\to 0}\|\mathcal{H}(\zeta,\varepsilon) - \mathcal{H}(\zeta,0)\|_1 = 0, \quad \lim_{\varepsilon\to 0}\|\mathrm{d}_1\mathcal{H}[\zeta,\varepsilon] - \mathrm{d}_1\mathcal{H}[\zeta,0]\|_{\mathcal{L}(H^1(\mathbb{R}))} = 0$$

uniformly over $\zeta \in B_R(0)$. The equation

$$\mathcal{H}(\zeta,0) = \zeta + 2c_0^2 d_0 \left(2c_0^2 - (\tfrac{9}{8} - \tfrac{1}{8}\gamma)\partial_Z^2\right)^{-1} \zeta^2 = 0$$

has the (unique) nontrivial solution ζ_{KdV} in $H^1_{\mathrm{e}}(\mathbb{R})$ and it remains to show that

$$\mathrm{d}_1\mathcal{H}[\zeta_{\mathrm{KdV}},0] = I + 4c_0^2 d_0 \left(2c_0^2 - (\tfrac{9}{8} - \tfrac{1}{8}\gamma)\partial_Z^2\right)^{-1} (\zeta_{\mathrm{KdV}}\cdot)$$

is an isomorphism. This result follows from the following lemma.

Lemma 4.4

(i) The formula $\zeta \mapsto 4c_0^2 d_0 \left(2c_0^2 - (\tfrac{9}{8} - \tfrac{1}{8}\gamma)\partial_Z^2\right)^{-1} (\zeta_{\mathrm{KdV}}\cdot)$ *defines a compact linear operator* $H^1(\mathbb{R}) \to H^1(\mathbb{R})$ *and* $H^1_{\mathrm{e}}(\mathbb{R}) \to H^1_{\mathrm{e}}(\mathbb{R})$*, and in particular* $\mathrm{d}_1\mathcal{H}[\zeta_{\mathrm{KdV}},0]$ *is a Fredholm operator with index* 0*.*

(ii) Every bounded solution of the equation

$$(\tfrac{1}{8}\gamma - \tfrac{9}{8})\zeta_{ZZ} + 2c_0^2\zeta + 4c_0^2 d_0 \zeta_{\mathrm{KdV}}\zeta = 0, \tag{65}$$

is a multiple of $\zeta_{\mathrm{KdV},Z}$ *and is therefore antisymmetric. In particular* $\ker \mathrm{d}_1\mathcal{H}[\zeta_{\mathrm{KdV}},0]$ *is trivial.*

Theorem 1.1 follows from Theorem 4.2 and the following result.

Proposition 4.5 *The formulae*

$$\eta = \eta_1 + \eta_2(\eta_1), \quad \eta_1(z) = \varepsilon^2\zeta_\varepsilon(\varepsilon z)$$

lead to the estimate

$$\eta(z) = \varepsilon^2\zeta_{\mathrm{KdV}}(\varepsilon z) + o(\varepsilon^2)$$

uniformly over $z \in \mathbb{R}$

Proof. Note that

$$\|\zeta_\varepsilon - \zeta_{\mathrm{KdV}}\|_\infty \lesssim \|\zeta_\varepsilon - \zeta_{\mathrm{KdV}}\|_1 = o(1),$$

so that

$$\eta_1(z) = \varepsilon^2\zeta_{\mathrm{KdV}}(\varepsilon z) + \varepsilon^2\big(\zeta_\varepsilon(\varepsilon z) - \zeta_{\mathrm{KdV}}(\varepsilon z)\big) = \varepsilon^2\zeta_{\mathrm{KdV}}(\varepsilon z) + o(\varepsilon^2)$$

uniformly over $z \in \mathbb{R}$. Furthermore

$$\|\eta_2(\eta_1)\|_\infty \lesssim \|\eta_2(\eta_1)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2 = \varepsilon^{7/2}\|\zeta_\varepsilon\|_1^2 \lesssim \varepsilon^{7/2}. \qquad \square$$

4.2 Weak surface tension

Theorem 4.6 *For each sufficiently small value of* $\varepsilon > 0$ *equation* (61) *has two small-amplitude, symmetric solutions* $\zeta_\varepsilon^\pm$ *in* $H^1_\varepsilon(\mathbb{R})$ *with* $\|\zeta_\varepsilon^\pm \mp \zeta_{\mathrm{NLS}}\|_1 \to 0$ *as* $\varepsilon \to 0$*.*

We again begin the proof of Theorem 4.6 by 'replacing' the nonlocal operator in the fixed-point formulation

$$\zeta + \varepsilon^2\big(\varepsilon^2 c_0^2 f(\omega) + g(\omega + \varepsilon D)\big)^{-1}\Big(-a_3\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{\mathcal{O}}_0^\varepsilon(\|\zeta\|_1)\Big) = 0 \tag{66}$$

of equation (61) for $\zeta \in H^1_\varepsilon(\mathbb{R})$ with a differential operator.

Proposition 4.7 *The inequality*

$$\left|\frac{\varepsilon^2}{c_0^2 f(\omega)\varepsilon^2 + g(\omega + \varepsilon k)} - \frac{1}{a_2 + a_1 k^2}\right| \lesssim \frac{\varepsilon}{(1+k^2)^{1/2}}$$

holds uniformly over $|k| < \delta/\varepsilon$*.*

Using the above proposition, one can write equation (66) as

$$\zeta + \tilde{F}_\varepsilon(\zeta) = 0,$$

where

$$\tilde{F}_\varepsilon(\zeta) = F_\varepsilon(\chi_0(\varepsilon D)\zeta), \qquad F_\varepsilon(\zeta) = -a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\chi_0(\varepsilon D)(|\zeta|^2\zeta) + \varepsilon^{1/2}\underline{\mathcal{O}}_1^\varepsilon(\|\zeta\|_1),$$

and establish Theorem 4.6 by applying Theorem 4.1 with

$$\mathcal{W} = H^1_{\mathrm{e}}(\mathbb{R},\mathbb{C}) = \{\zeta \in H^1(\mathbb{R}) : \zeta(Z) = \overline{\zeta(-Z)} \text{ for all } Z \in \mathbb{R}\},$$

$W_0 = B_R(0)$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0 and

$$\mathcal{H}(\zeta,\varepsilon) := \zeta + \tilde{F}_{|\varepsilon|}(\zeta).$$

Observe that

$$\begin{aligned}
&\mathcal{H}(\zeta,\varepsilon) - \mathcal{H}(\zeta,0)\\
&\quad = -a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\Big[\chi_0(|\varepsilon|D)\big(|\chi_0(|\varepsilon|D)\zeta|^2(\chi_0(|\varepsilon|D) - I)\zeta + |\zeta|^2(\chi_0(|\varepsilon|D) - I)\zeta\\
&\qquad\qquad + \zeta\chi_0(|\varepsilon|D)\zeta(\chi_0(|\varepsilon|D) - I)\bar{\zeta}\big)\\
&\qquad\qquad + (\chi_0(|\varepsilon D|) - I)|\zeta|^2\zeta\Big] + |\varepsilon|^{\frac{1}{2}}\underline{\mathcal{O}}_1^{|\varepsilon|}(\|\zeta\|_1);
\end{aligned}$$

noting that $H^1(\mathbb{R};\mathbb{C})$ is a Banach algebra, that $\chi_0(|\varepsilon|D) \to I$ in $\mathcal{L}(H^1(\mathbb{R}), H^{3/4}(\mathbb{R})) = 0$ as $\varepsilon \to 0$ and that pointwise multiplication defines a bounded trilinear mapping $(H^1(\mathbb{R};\mathbb{C})^2 \times H^{3/4}(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ (see Hörmander [14, Theorem 8.3.1]), one concludes that

$$\lim_{\varepsilon\to 0}\|\mathcal{H}(\zeta,\varepsilon) - \mathcal{H}(\zeta,0)\|_1 = 0, \quad \lim_{\varepsilon\to 0}\|\mathrm{d}_1\mathcal{H}[\zeta,\varepsilon] - \mathrm{d}_1\mathcal{H}[\zeta,0]\|_{\mathcal{L}(H^1(\mathbb{R},\mathbb{C}))} = 0$$

uniformly over $\zeta \in B_R(0)$.

The equation

$$\mathcal{H}(\zeta,0) = \zeta - a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}|\zeta|^2\zeta = 0$$

has (precisely two) nontrivial solutions $\pm\zeta_{\mathrm{NLS}}$ in $H^1_{\mathrm{e}}(\mathbb{R},\mathbb{C})$, which are both real, and the fact that $\mathrm{d}_1\mathcal{H}[\pm\zeta_{\mathrm{NLS}},0]$ is an isomorphism is conveniently established by using real coordinates. Define $\zeta_1 = \operatorname{Re}\zeta$ and $\zeta_2 = \operatorname{Im}\zeta$, so that

$$\mathrm{d}_1\mathcal{H}[\pm\zeta_{\mathrm{NLS}},0](\zeta_1 + \mathrm{i}\zeta_2) = \mathcal{H}_1(\zeta_1) + \mathrm{i}\mathcal{H}_2(\zeta_2),$$

where $\mathcal{H}_1 : H^1_{\mathrm{e}}(\mathbb{R}) \to H^1_{\mathrm{e}}(\mathbb{R})$ and $\mathcal{H}_2 : H^1_{\mathrm{o}}(\mathbb{R}) \to H^1_{\mathrm{o}}(\mathbb{R})$ are given by

$$\mathcal{H}_1(\zeta_1) = \zeta_1 - 3a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\zeta_{\mathrm{NLS}}^2\zeta_1, \qquad \mathcal{H}_2(\zeta_2) = \zeta_2 - a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\zeta_{\mathrm{NLS}}^2\zeta_2$$

and

$$\begin{aligned}
H^1_{\mathrm{e}}(\mathbb{R}) &:= \{u \in H^1(\mathbb{R}) : u(Z) = u(-Z) \text{ for all } Z \in \mathbb{R}\},\\
H^1_{\mathrm{o}}(\mathbb{R}) &:= \{u \in H^1(\mathbb{R}) : u(Z) = -u(-Z) \text{ for all } Z \in \mathbb{R}\}.
\end{aligned}$$

Proposition 4.8

(i) The formulae

$$\zeta_1 \mapsto -3a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\zeta_{\mathrm{NLS}}^2\zeta_1, \qquad \zeta_2 \mapsto -a_3\left(a_2 - a_1\partial_Z^2\right)^{-1}\zeta_{\mathrm{NLS}}^2\zeta_2$$

define compact linear operators $H^1(\mathbb{R}) \to H^1(\mathbb{R})$, $H^1_{\mathrm{e}}(\mathbb{R}) \to H^1_{\mathrm{e}}(\mathbb{R})$ and $H^1_{\mathrm{o}}(\mathbb{R}) \to H^1_{\mathrm{o}}(\mathbb{R})$, and in particular $\mathcal{H}_1$, $\mathcal{H}_2$ are Fredholm operators with index 0.

(ii) Every bounded solution of the equation

$$-a_1\zeta_{1ZZ} + a_2\zeta_1 - 3a_3\zeta_{\mathrm{NLS}}^2\zeta_1 = 0 \tag{67}$$

is a multiple of $\zeta_{\mathrm{NLS},Z}$ *and is therefore antisymmetric, while every bounded solution of the equation*

$$-a_1\zeta_{1ZZ} + a_2\zeta_1 - a_3\zeta_{\mathrm{NLS}}^2\zeta_1 = 0 \tag{68}$$

is a multiple of ζ_{NLS} *and is therefore symmetric. In particular* $\ker\mathcal{H}_1$ *and* $\ker\mathcal{H}_2$ *are trivial.*

Theorem 1.2 follows from Theorem 4.6 and the following result.

Proposition 4.9 *The formulae*

$$\eta = \eta_1 + F(\eta_1) + \eta_3(\eta_1), \quad \eta_1 = \eta_1^+ + \eta_1^-, \quad \eta_1^+(z) = \tfrac{1}{2}\varepsilon\zeta_\varepsilon^\pm(\varepsilon z)\mathrm{e}^{\mathrm{i}\omega z}$$

leads to the estimate

$$\eta(z) = \pm\varepsilon\zeta_{\mathrm{NLS}}(\varepsilon z)\cos(\omega z) + o(\varepsilon)$$

uniformly over $z \in \mathbb{R}$.

Proof. Note that

$$\|\zeta_\varepsilon \mp \zeta_{\mathrm{NLS}}\|_\infty \lesssim \|\zeta_\varepsilon \mp \zeta_{\mathrm{NLS}}\|_1 = o(1),$$

so that

$$\eta_1^+(z) = \pm\tfrac{1}{2}\varepsilon\zeta_{\mathrm{NLS}}(\varepsilon z)\mathrm{e}^{\mathrm{i}\omega z} + \tfrac{1}{2}\varepsilon\big(\zeta_\varepsilon^\pm(\varepsilon z) \mp \zeta_{\mathrm{NLS}}(\varepsilon z)\big)\mathrm{e}^{\mathrm{i}\omega z} = \pm\tfrac{1}{2}\varepsilon\zeta_{\mathrm{NLS}}(\varepsilon z)\mathrm{e}^{\mathrm{i}\omega z} + o(\varepsilon)$$

uniformly over $z \in \mathbb{R}$. Furthermore

$$\|F(\eta_1)\|_\infty \lesssim \|F(\eta_1)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2 = \varepsilon^{3/2}\|\zeta_\varepsilon^\pm\|_1^2 \lesssim \varepsilon^{3/2}$$

and

$$\|\eta_3(\eta_1)\|_\infty \lesssim \|\eta_3(\eta_1)\|_2 \lesssim \varepsilon|||\eta_1|||^2 = \varepsilon^2\|\zeta_\varepsilon^\pm\|_1^2 \lesssim \varepsilon^2. \qquad \square$$

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Appendix A Dispersion relation

In this appendix we establish the qualitative features of the dispersion relation

$$c^2 = \frac{\gamma - 1 + k^2}{f(k)}$$

shown in Figure 4. Note that $c^2(0) = \frac{1}{2}(\gamma - 1)$ and $c^2(k) \to \infty$ as $k \to \infty$. Furthermore, the calculation

$$\frac{\mathrm{d}c^2}{\mathrm{d}k}(k) = \frac{2kf(k) - (\gamma - 1 + k^2)f'(k)}{f(k)^2}, \qquad f'(k) = k - \frac{kI_0(k)I_2(k)}{I_1(k)^2} \tag{69}$$

shows that

$$\frac{\mathrm{d}c^2}{\mathrm{d}k}(0) = 0,$$

and it remains to determine whether c^2 has any critical points at positive values of k.

Proposition A.1 *The function*

$$h(k) = 1 - k^2 + \frac{2kf(k)}{f'(k)}, \qquad k \geq 0,$$

is strictly monotone increasing.

Proof. Observe that

$$h'(k) = -2k + 2f(k)\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{k}{f'(k)}\right) + 2k = -2f(k)\big(\phi_1(k)\big)^{-2}\phi_1'(k), \tag{70}$$

where

$$\phi_1(k) := \frac{1}{k}f'(k) = 1 - \frac{I_0(k)I_2(k)}{I_1^2(k)}.$$

Barciz [1, p. 257] showed that for each $\nu > -1$ the function

$$\phi_\nu(k) = 1 - \frac{I_{\nu-1}(k)I_{\nu+1}(k)}{I_\nu^2(k)}, \qquad k \geq 0,$$

satisfies $\phi_\nu'(k) < 0$ for $k > 0$ with $\phi_\nu'(0) = 0$. It follows from equation (70) that $h'(k) > 0$ for $k > 0$ with $h'(0) = 0$, so that h is strictly monotone increasing (note that $\phi_1(k) > 0$ since $\phi_1(0) = \frac{1}{2}$, $\phi_1(k) \to 0$ as $k \to \infty$ and ϕ_1 is strictly monotone decreasing). □

Observing that $h(0) = 9$ and $h(k) \to \infty$ as $k \to \infty$, we find from (69) that for each fixed $\gamma > 9$ there exists a unique $\omega > 0$ with

$$\gamma = 1 - \omega^2 + \frac{2\omega f(\omega)}{f'(\omega)}, \qquad \frac{\mathrm{d}c^2}{\mathrm{d}k}(\omega) = 0,$$

while c^2 has no critical points at positive values of k for $1 < \gamma \leq 9$. It follows that c^2 is a strictly monotone increasing function of k for $1 < \gamma \leq 9$, while for $\gamma > 9$ it has a unique local maximum at $k = 0$ and a unique global minimum at $k = \omega > 0$, where $\omega = h^{-1}(\gamma) > 0$.

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Appendix B Weakly nonlinear theory

Formal derivation of the KdV equation for $1 < \gamma < 9$

We choose

$$c_0^2 = \tfrac{1}{2}(\gamma - 1),$$

write $c^2 = c_0^2(1 - \varepsilon^2)$ and substitute the Ansatz

$$\eta(z) = \varepsilon^2\zeta_1(Z) + \varepsilon^4\zeta_2(Z) + \cdots, \qquad Z = \varepsilon z,$$

into equation (15). Expanding

$$\begin{aligned} K_0 &= f(\varepsilon D) \\ &= \underbrace{f(0)}_{=2} - \tfrac{1}{2}\varepsilon^2 \underbrace{f''(0)}_{=\frac{1}{2}} \partial_Z^2 + O(\varepsilon^4), \end{aligned}$$

where $D = -\mathrm{i}\partial_Z$, we find from Corollary 2.16 that

$$\begin{aligned} \mathcal{K}_1(\eta) &= \varepsilon^2(\gamma - 1)\zeta_1 + \varepsilon^4\big(-\zeta_{1ZZ} + (\gamma - 1)\zeta_2\big) + O(\varepsilon^6), \\ \mathcal{K}_2(\eta) &= \varepsilon^4(-\gamma - \tfrac{1}{2}\gamma\nu''(1) + 1)\zeta_1^2 + O(\varepsilon^6), \end{aligned}$$

$$\begin{aligned} \mathcal{L}_1(\eta) &= 2\varepsilon^2\zeta_1 + \varepsilon^4(-\tfrac{1}{4}\zeta_{1ZZ} + 2\zeta_2) + O(\varepsilon^6), \\ \mathcal{L}_2(\eta) &= -5\varepsilon^4\zeta_1^2 + O(\varepsilon^6) \end{aligned}$$

and of course $\mathcal{K}_j(\eta), \mathcal{L}_j(\eta) = O(\varepsilon^6)$ for $j \geq 3$.

The $O(\varepsilon^2)$ component of equation (15) is trivially satisfied, while the $O(\varepsilon^4)$ component yields the KdV equation

$$(\tfrac{1}{8}\gamma - \tfrac{9}{8})\zeta_{1ZZ} + 2c_0^2\zeta_1 + 2c_0^2 d_0\zeta_1^2 = 0.$$

Formal derivation of the NLS equation for $\gamma > 9$

We choose

$$\gamma = 1 - \omega^2 + \frac{2\omega f(\omega)}{f'(\omega)}, \qquad c_0^2 = \frac{2\omega}{f'(\omega)},$$

write $c^2 = c_0^2(1-\varepsilon^2)$ and substitute the Ansatz

$$\eta(z) = \varepsilon\eta_1(z,Z) + \varepsilon^2\eta_2(z,Z) + \varepsilon^3\eta_3(z,Z) + \cdots, \qquad Z = \varepsilon z,$$

into equation (15). Expanding

$$\begin{aligned} K_0 &= f(d+\varepsilon D) \\ &= f(d) - \mathrm{i}\varepsilon f'(d)\partial_Z - \tfrac{1}{2}\varepsilon^2 f''(d)\partial_Z^2 + O(\varepsilon^3), \end{aligned}$$

where $d = -\mathrm{i}\partial_z$, $D = -\mathrm{i}\partial_Z$, we find from Corollary 2.16 that

$$\begin{aligned} \mathcal{K}_1(\eta) &= \varepsilon\big((\gamma-1)\eta_1 - \eta_{1zz}\big) + \varepsilon^2\big((\gamma-1)\eta_2 - \eta_{2zz} - 2\eta_{1zZ}\big) \\ &\quad + \varepsilon^3\big((\gamma-1)\eta_3 - \eta_{3zz} - 2\eta_{2zZ} - \eta_{1ZZ}\big) + O(\varepsilon^4), \\ \mathcal{K}_2(\eta) &= \varepsilon^2(A_0\eta_1^2 - \tfrac{1}{2}\eta_{1z}^2) + \varepsilon^3(2A_0\eta_1\eta_2 - \eta_{1z}\eta_{1Z} - \eta_{1z}\eta_{2z}) + O(\varepsilon^4), \\ \mathcal{K}_3(\eta) &= \varepsilon^3(B_0\eta_1^3 + \tfrac{1}{2}\eta_1\eta_{1z}^2 + \tfrac{3}{2}\eta_{1z}^2\eta_{1zz}) + O(\varepsilon^4), \end{aligned}$$

$$\begin{aligned} \mathcal{L}_1(\eta) &= \varepsilon f(d)\eta_1 + \varepsilon^2(f(d)\eta_2 - \mathrm{i}f'(d)\eta_{1Z}) \\ &\quad + \varepsilon^3(f(d)\eta_3 - \mathrm{i}f'(d)\eta_{2Z} - \tfrac{1}{2}f''(d)\eta_{1ZZ}) + O(\varepsilon^4), \\ \mathcal{L}_2(\eta) &= \varepsilon^2\big(-\tfrac{1}{2}\eta_{1z}^2 - \tfrac{1}{2}(f(d)\eta_1)^2 - \eta_{1zz}\eta_1 - f(d)(\eta_1 f(d)\eta_1) + \tfrac{1}{2}f(d)\eta_1^2\big) \\ &\quad + \varepsilon^3\big(-\eta_{1z}\eta_{1Z} - \eta_{1z}\eta_{2z} - \eta_{1zz}\eta_2 - \eta_1\eta_{2zz} - 2\eta_1\eta_{1zZ} - (f(d)\eta_1)(f(d)\eta_2) \\ &\qquad + \mathrm{i}(f(d)\eta_1)(f'(d)\eta_{1Z}) + \mathrm{i}f(d)(\eta_1 f'(d)\eta_{1Z}) + \mathrm{i}f'(d)(\eta_1 f(d)\eta_1)_Z \\ &\qquad - f(d)(\eta_1 f(d)\eta_2) - f(d)(\eta_2 f(d)\eta_1) - \tfrac{1}{2}\mathrm{i}f'(d)(\eta_1^2)_Z + f(d)(\eta_1\eta_2)\big) + O(\varepsilon^4), \\ \mathcal{L}_3(\eta) &= \varepsilon^3\big(-\tfrac{1}{2}(\eta_1^2\eta_{1z})_z + \tfrac{1}{2}(f(d)\eta_1)(\eta_1^2)_{zz} + (f(d)\eta_1)(f(d)(\eta_1 f(d)\eta_1)) \\ &\quad - \tfrac{1}{2}(f(d)\eta_1)(f(d)\eta_1^2) - (f(d)\eta_1)\eta_{1z}^2 + \tfrac{1}{2}(\eta_1^2 f(d)\eta_1)_{zz} + \tfrac{1}{2}f(d)(\eta_1^2\eta_{1zz}) \\ &\quad - \tfrac{1}{2}f(d)(\eta_1^2 f(d)\eta_1) + f(d)(\eta_1 f(d)(\eta_1 f(d)\eta_1)) - \tfrac{1}{2}f(d)(\eta_1 f(d)\eta_1^2)\big) + O(\varepsilon^4), \end{aligned}$$

and of course $\mathcal{K}_j(\eta), \mathcal{L}_j(\eta) = O(\varepsilon^4)$ for $j \geq 4$.

The next step is to substitute the expressions

$$\begin{aligned} \eta_1(z,Z) &= \zeta_1(Z)\mathrm{e}^{\mathrm{i}\omega z} + \text{c.c.}, \\ \eta_2(z,Z) &= \zeta_0(Z) + \zeta_2(Z)\mathrm{e}^{2\mathrm{i}\omega z} + \text{c.c.}, \\ \eta_3(z,Z) &= \zeta_6(Z) + \zeta_5(Z)\mathrm{e}^{\mathrm{i}\omega z} + \zeta_4(Z)\mathrm{e}^{2\mathrm{i}\omega z} + \zeta_3(Z)\mathrm{e}^{3\mathrm{i}\omega z} + \text{c.c.}, \end{aligned}$$

into the previous expansions. Noting that

$$f^{(j)}(d)(\mathrm{e}^{\mathrm{i}n\omega z}) = f^{(j)}(n\omega)\mathrm{e}^{\mathrm{i}n\omega z}, \qquad j \in \mathbb{N}_0,\ n \in \mathbb{Z},$$

we find that the $O(\varepsilon)$ component of equation (15) is

$$g(\omega)\zeta_1\mathrm{e}^{\mathrm{i}\omega z} + \text{c.c} = 0,$$

which is satisfied because $g(\omega) = 0$. The $O(\varepsilon^2)$ component yields the equation

$$\begin{aligned} &g(0)\zeta_0 + (2A_0 - \omega^2 - c_0^2 B(\omega))|\zeta_1|^2 + \mathrm{i}g'(\omega)\zeta_1'\mathrm{e}^{\mathrm{i}\omega z} \\ &\qquad + \big(g(2\omega)\zeta_2 + (A_0 + \tfrac{1}{2}\omega^2 - c_0^2 A(\omega))\zeta_1^2\big)\mathrm{e}^{2\mathrm{i}\omega z} + \text{c.c.} = 0, \end{aligned}$$

where

$$A(\omega) = \tfrac{3}{2}\omega^2 - \tfrac{1}{2}f(\omega)^2 - f(\omega)f(2\omega) + \tfrac{1}{2}f(2\omega), \qquad B(\omega) = \omega^2 - f(\omega)^2 - 4f(\omega) + 2;$$

since $g'(\omega)=0$ this equation is satisfied by choosing

$$\begin{aligned}\zeta_0 &= g(0)^{-1}(\omega^2-2A_0+c_0^2B(\omega))|\zeta_1|^2,\\ \zeta_2 &= g(2\omega)(-\omega_0-\tfrac{1}{2}\omega^2+c_0^2A(\omega))\zeta_1^2.\end{aligned}$$

The coefficient of $\mathrm{e}^{\mathrm{i}\omega z}$ in the $O(\varepsilon^3)$ component of (15) yields the equation

$$\begin{aligned}g(\omega)\zeta_5 &+ 2(A_0-\omega^2-c_0^2C(\omega))\overline{\zeta}_1\zeta_2-\tfrac{1}{2}g''(\omega)\zeta_1''+c_0^2f(\omega)\zeta_1\\ &+2(A_0-c_0^2D(\omega))\zeta_0\zeta_1+(3B_0+\tfrac{1}{2}\omega^2-\tfrac{3}{2}\omega^4-c_0^2E(\omega))|\zeta_1|\zeta_1|^2=0,\end{aligned}\tag{71}$$

where

$$\begin{aligned}C(\omega) &= \tfrac{3}{2}\omega^2-f(\omega)f(2\omega)+\tfrac{1}{2}f(\omega)-\tfrac{1}{2}f(\omega)^2,\\ D(\omega) &= \tfrac{1}{2}\omega^2-\tfrac{3}{2}f(\omega)-\tfrac{1}{2}f(\omega)^2,\\ E(\omega) &= 2f(\omega)^2f(2\omega)-6f(\omega)\omega^2+\tfrac{13}{2}f(\omega)^2-f(\omega)f(2\omega)-4f(\omega)+\tfrac{1}{2}\omega^2.\end{aligned}$$

Substituting for ζ_0 and ζ_2 into equation (71) and setting $g(\omega)=0$ yields the nonlinear Schrödinger equation

$$-a_1\zeta_{1ZZ}+a_2\zeta-a_3|\zeta_1|^2\zeta_1=0,$$

where $a_1=\frac{1}{2}g''(\omega)$, $a_2=c_0^2f(\omega)$ and

$$\begin{aligned}4a_3 = 2g(2\omega)^{-1}&(c_0^2C(\omega)-A_0+\omega^2)(c_0^2A(\omega)-A_0-\tfrac{1}{2}\omega^2)\\ &+2g(0)^{-1}(c_0^2D(\omega)-A_0)(c_0^2B(\omega)-2A_0+\omega^2)\\ &-3B_0-\tfrac{1}{2}\omega^2+\tfrac{3}{2}\omega^4+c_0^2E(\omega).\end{aligned}$$

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