Fully localised three-dimensional solitary water waves on Beltrami flows with strong surface tension

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Abstract

Fully localised three-dimensional solitary waves are steady water waves which are evanescent in every horizontal direction. This paper presents an existence theory for such waves under the assumptions that the relative vorticity and velocity fields are parallel ('Beltrami flows'), that the free surface of the water takes the form $\{z = \eta(x, y)\}$ for some function $\eta : \mathbb{R}^2 \to \mathbb{R}$, and that the influence of surface tension is sufficiently strong. The governing equations are formulated as a single equation for η, which is then reduced to a perturbation of the KP-I equation. This equation has recently been shown to have a family of nondegenerate localised solutions, and an application of a suitable variant of the implicit-function theorem shows that they persist under perturbations.

1 Introduction

1.1 The hydrodynamic problem

Consider an incompressible perfect fluid of unit density occupying a three-dimensional domain bounded below by a rigid horizontal plane and above by a free surface. A *steady water wave* is a fluid flow of this kind in which both the velocity field and free-surface profile are stationary with respect to a uniformly (horizontally) translating frame of reference; a *(fully localised) solitary wave* is a nontrivial steady wave whose free surface decays to the height of the fluid at rest in every horizontal direction. Working in frame of reference moving with the wave and in dimensionless coordinates, we suppose that the fluid domain is $D_\eta = \{(x, y, z) : -1 < z < \eta(x, y)\}$ (so that the free surface is the graph S_η of an unknown function $\eta : \mathbb{R}^2 \to \mathbb{R}$), and the flow is a *(strong) Beltrami flow* whose velocity and vorticity fields $\boldsymbol{u}$ and $\operatorname{curl}\boldsymbol{u}$ are parallel, so that $\operatorname{curl}\boldsymbol{u} = \alpha\boldsymbol{u}$ for some fixed constant α. *Irrotational* flows (with $\operatorname{curl}\boldsymbol{u} = \boldsymbol{0}$) are included as the special case $\alpha = 0$. The hydrodynamic problem is to solve the equations

$$\operatorname{curl}\boldsymbol{u} = \alpha\boldsymbol{u} \qquad \text{in } D_\eta, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } D_\eta, \tag{2}$$

$$\boldsymbol{u}\cdot\boldsymbol{e}_3 = 0 \qquad \text{at } z = -1, \tag{3}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{at } z = \eta, \tag{4}$$

$$\tfrac{1}{2}|\boldsymbol{u}|^2 + \eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_y = \tfrac{1}{2}|\boldsymbol{c}|^2 \qquad \text{at } z = \eta, \tag{5}$$

where $\nabla = (\partial_x, \partial_y)^T$, $\nabla^\perp = (\partial_y, -\partial_x)^T$, $\boldsymbol{c} := (c_1, c_2)^T$ is the dimensionless wave velocity, $\boldsymbol{e}_3 = (0, 0, 1)^T$ and $\boldsymbol{n} := (-\eta_x, -\eta_y, 1)^T$ is the outward normal vector at S_η; we have also introduced the Bond number $\beta = \sigma/gh^2$, where h is the depth of the fluid at rest, g is the acceleration due to gravity and $\sigma > 0$ is the coefficient of surface tension. (The pressure p in the fluid is recovered using the formula $p(x, y, z) = -\frac{1}{2}|\boldsymbol{u}(x, y, z)|^2 - y$, and the variables $\boldsymbol{u}$ and p automatically solve the stationary Euler equation in D_η.) Equations (4) and (5) are referred to as respectively the *kinematic* and *dynamic* boundary conditions at the free surface. It is natural to write η and $\boldsymbol{u}$ as a perturbations of the trivial solution

$$\eta^\star = 0, \qquad \boldsymbol{u}^\star = c_1\begin{pmatrix}\cos\alpha z\\ -\sin\alpha z\\ 0\end{pmatrix} + c_2\begin{pmatrix}\sin\alpha z\\ \cos\alpha z\\ 0\end{pmatrix} \tag{6}$$

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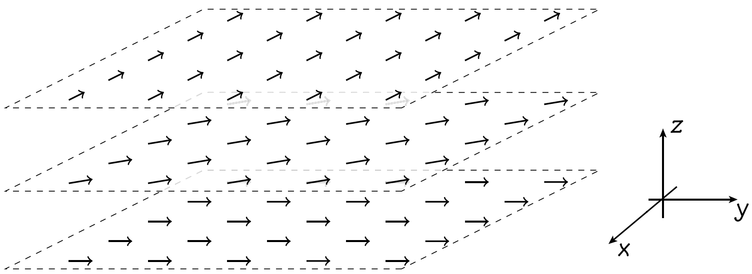

Figure 1: *The trivial flow* (6)

of (1)–(5) (see Figure 1), so that $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^\star$ satisfies the equations

$$\operatorname{curl} \boldsymbol{v} = \alpha \boldsymbol{v} \qquad \text{in } D_\eta, \qquad (7)$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } D_\eta, \qquad (8)$$

$$\boldsymbol{v} \cdot \boldsymbol{e}_3 = 0 \qquad \text{at } z = -1, \qquad (9)$$

$$\boldsymbol{v} \cdot \boldsymbol{n} + \boldsymbol{u}^\star \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta, \qquad (10)$$

$$\tfrac{1}{2}|\boldsymbol{v}|^2 + \boldsymbol{v} \cdot \boldsymbol{u}^\star + \eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_y = 0 \qquad \text{at } z = \eta. \qquad (11)$$

Our task is to find solutions $(\eta, \boldsymbol{v})$ of (7)–(11) which are evanescent as $|(x, y)| \to \infty$ and therefore represent fully localised solitary waves 'riding' the trivial flow (6). Note that these equations are invariant under

$$\eta(x,y) \mapsto \eta(-x,-y), \qquad (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)) \mapsto (v_1(-x,-y,z), v_2(-x,-y,z), -v_3(-x,-y,z)),$$

and we in fact seek solutions which are themselves invariant under this transformation.

Irrotational fully localised solitary waves have been found by Groves & Sun [13] and Buffoni *et al.* [1] for $\beta > \frac{1}{3}$ using variational methods. Their result has recently been made more precise by Gui *et al.* [14, 15], who obtained waves which are perturbations and scalings of localised 'lump' solutions of the KP-I equation and discussed their stability. In this paper we use a related method to obtain the same family of waves for equations (7)–(10) for sufficiently large values of β (depending upon α); the existence result of Gui *et al.* is included as a special case. Other types of three-dimensional steady water waves have also been studied, in particular *doubly periodic* steady waves, that is waves which are periodic in two different horizontal directions. Their existence was established for irrotational flows with surface tension ($\beta > 0$) by Craig & Nicholls [4, 5], for irrotational waves without surface tension ($\beta = 0$) by Iooss & Plotnikov [16] and for Beltrami flows with surface tension by Lokharu, Seth & Wahlén [20] (see also Groves *et al.* [12] for an existence theory in a framework similar to that used in the present paper). Doubly periodic gravity-capillary waves with more general (but small) vorticity were recently constructed by Seth, Varholm & Wahlén [23].

1.2 Heuristics

The KP-I equation arises in weakly nonlinear theory as a universal model equation for two-dimensional nonlinear dispersive systems whose linearisation has a distinguished wave speed attained only by long waves. A straightforward calculation shows that the linearised version of (7)–(11) has solutions of the form $\eta(x,y) \sim \cos(\boldsymbol{k} \cdot \boldsymbol{x})$, where $\boldsymbol{x} = (x, y)^{\mathrm{T}}$ and $\boldsymbol{k} = (k_1, k_2)^{\mathrm{T}}$, if

$$g(\boldsymbol{k}) = 0, \qquad (12)$$

where

$$g(\boldsymbol{k}) = -\frac{1}{|\boldsymbol{k}|^2}(\alpha(\boldsymbol{c} \cdot \boldsymbol{k}^\perp)(\boldsymbol{c} \cdot \boldsymbol{k}) + \mathsf{c}(|\boldsymbol{k}|^2)(\boldsymbol{c} \cdot \boldsymbol{k})^2) + 1 + \beta|\boldsymbol{k}|^2$$

and

$$\mathsf{c}(\mu) = \begin{cases} \sqrt{\alpha^2 - \mu}\, \cot(\sqrt{\alpha^2 - \mu}), & \text{if } \mu < \alpha^2, \\ \sqrt{\mu - \alpha^2}\, \coth(\sqrt{\mu - \alpha^2}), & \text{if } \mu \geq \alpha^2. \end{cases}$$

The calculation

$$g(\boldsymbol{k}) = -\frac{1}{1+\frac{k_2^2}{k_1^2}}\alpha\big(c_1\tfrac{k_2}{k_1}-c_2\big)\big(c_1+c_2\tfrac{k_2}{k_1}\big) - \frac{1}{1+\frac{k_2^2}{k_1^2}}\mathsf{c}\big(k_1^2(1+\tfrac{k_2^2}{k_1^2})\big)\big(c_1+c_2\tfrac{k_2}{k_1}\big)^2 + 1 + \beta k_1^2(1+\tfrac{k_2^2}{k_1^2})$$

shows that g is an analytic function $\tilde{g}$ of k_1 and $\frac{k_2}{k_1}$. We find that $\tilde{g}(0,0)=0$ if $\boldsymbol{c}=\boldsymbol{c}_0$, where

$$\boldsymbol{c}_0 = \begin{pmatrix} c_0\cos\frac{1}{2}\alpha \\ -c_0\sin\frac{1}{2}\alpha \end{pmatrix}, \qquad c_0^2 = \tfrac{2}{\alpha}\tan\tfrac{1}{2}\alpha,$$

and it is shown in Appendix A that the dispersion relation (12) has no further solutions for sufficiently large values of β. Substituting the Ansatz

$$\boldsymbol{c} = (1-\varepsilon^2)\boldsymbol{c}_0$$

and

$$\eta(x,y) = \varepsilon^2\zeta(X,Y), \qquad X=\varepsilon x,\ Y=\varepsilon^2 y \tag{13}$$

into equations (7)–(11), one duly finds that to leading order ζ satisfies the KP-I equation

$$-(\beta-\beta_0)\zeta_{xx} + 2\zeta + \sec^2\tfrac{1}{2}\alpha\frac{D_2^2}{D_1^2}\zeta + d_\alpha\zeta^2 = 0, \tag{14}$$

where

$$\beta_0 = \frac{1}{2\alpha^2}(-\cos\alpha + \alpha\operatorname{cosec}\alpha), \qquad d_\alpha = \alpha\operatorname{cosec}\alpha + \tfrac{1}{2}\alpha\cot\alpha,$$

we have replaced (X,Y) with (x,y) for notational simplicity and $D_1=-\mathrm{i}\partial_x$, $D_2=-\mathrm{i}\partial_y$.

Equation (14) can be written in the normalised form

$$\partial_x^2(-\partial_x^2 u + u + 3u^2) + \partial_y^2 u = 0, \tag{15}$$

which has a family of explicit symmetric 'lump' solutions of the form

$$u_k^\star(x,y) = -2\partial_x^2\log\tau_k^\star(x,y), \qquad k=1,2,\ldots, \tag{16}$$

where $\tau_k^\star$ is a polynomial of degree $k(k+1)$ with $\tau_k^\star(x,y)=\tau_k(-x,y)=\tau_k^\star(x,-y)$ for all $(x,y)\in\mathbb{R}^2$; the first two members of the family are

$$\tau_1^\star(x,y) = x^2+y^2+3,$$
$$\tau_2^\star(x,y) = x^6+3x^4y^2+3x^2y^4+y^6+25x^4+90x^2y^2+17y^4-125x^2+475y^2+1875.$$

Note that the lump solutions $u_k^\star$ are smooth, decaying rational functions, so that the same is true of their derivatives of all orders. The functions $\zeta_1^\star$ and $\zeta_2^\star$ (where $\zeta_k^\star$ is obtained from $u_k^\star$ by reversing the normalisation) are sketched in Figure 2.

The following result was established by Liu & Wei [17] and Liu, Wei & Yang [19, 18].

Lemma 1.1

(i) Every smooth, algebraically decaying lump solution of (15) *has the form* $u(x,y)=-2\partial_x^2\log\tau(x,y)$ *for some polynomial* τ *of degree* $k(k+1)$ *with* $k\in\mathbb{N}$ *and satisfies* $|u(x,y)|\lesssim(1+x^2+y^2)^{-1}$ *for all* $(x,y)\in\mathbb{R}^2$.

(ii) There is a unique symmetric lump solution of the form (16) *for each* $k\in\mathbb{N}$ *with* $k(k+1)\leq 600$ *(and it is conjectured that this result holds for all* $k\in\mathbb{N}$*).*

(iii) The lump solutions $\zeta_1^\star$*,* $\zeta_2^\star$ *of* (15) *are nondegenerate in the sense that the only smooth, evanescent solutions of the linearised equation*

$$\partial_x^2(-\partial_x^2 u + u + 6u_k^\star u) + \partial_y^2 u = 0$$

for $k=1$*,* 2 *are linear combinations of* $\partial_x u_k^\star$ *and* $\partial_y u_k^\star$ *(and it is conjectured that this result holds for all* $k\in\mathbb{N}$*; see Remark 1.3 below).*

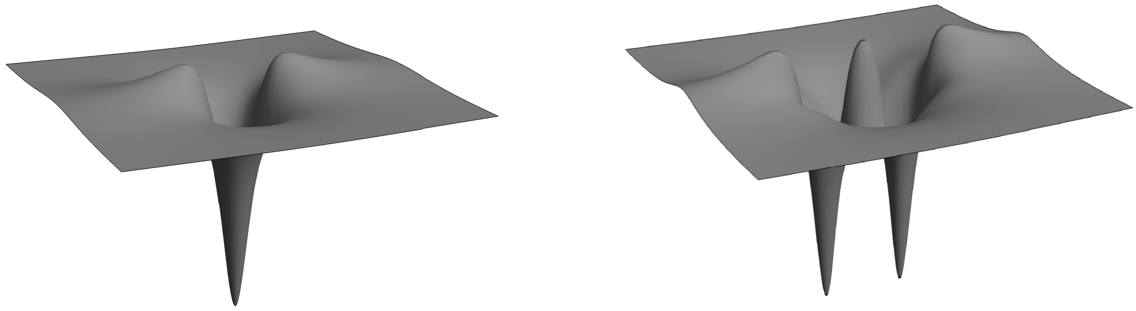

Figure 2: The KP lumps $\zeta_1^\star$ (left) and $\zeta_2^\star$ (right).

The KP lump solution ζ_k formally corresponds to a fully localised solitary water wave $\eta(x,z)=\varepsilon^2\zeta_k(\varepsilon x,\varepsilon^2 y)$. In this article we rigorously reduce the hydrodynamic equations (7)–(11) to a perturbation of the KP-I equation and combine the nondegeneracy result in Lemma 1.1(iii) with an implicit-function theorem argument to establish the following result.

Theorem 1.2 *Suppose that*

$$c_1 = c_0(1-\varepsilon^2)\cos\tfrac{1}{2}\alpha, \qquad c_2 = -c_0(1-\varepsilon^2)\sin\tfrac{1}{2}\alpha$$

with

$$c_0^2 = \tfrac{2}{\alpha}\tan\tfrac{1}{2}\alpha.$$

For each sufficiently large value of $\beta>0$ and each sufficiently small value of $\varepsilon>0$ equations (7)–(11) *possess fully localised solitary-wave solutions $\eta_1^\star$, $\eta_2^\star \in H^3(\mathbb{R}^2)$ which satisfy $\eta_k^\star(x,y)=\eta_k^\star(-x,-y)$ for all $(x,y)\in\mathbb{R}^2$ and*

$$\eta_k^\star(x,y)=\varepsilon^2\zeta_k^\star(\varepsilon x,\varepsilon^2 y)+o(\varepsilon^2) \tag{17}$$

uniformly over $(x,y)\in\mathbb{R}^2$.

Remark 1.3 *In fact Theorem 1.2 generates a fully localised solitary water wave from any symmetric lump solution $\zeta_k^\star$ of* (14) *which is nondegenerate in the sense of Lemma 1.1(iii), and a sketch of the proof of the nondegeneracy of $\zeta_k^\star$ for $k\geq 3$ was given by Liu, Wei & Yang [18].*

1.3 Reformulation

We proceed using a recent formulation of (7)–(11) due to Groves & Horn [11] which generalises the Zakharov-Craig-Sulem formulation of the irrotational problem (Zakharov [25], Craig & Sulem [6]). Let $\boldsymbol{F}_\parallel$ denote the horizontal component of the tangential part of a vector field $\boldsymbol{F}=(F_1,F_2,F_3)^{\mathrm{T}}$ at the free surface, so that $\boldsymbol{F}_\parallel=\boldsymbol{F}_\mathrm{h}+F_3\nabla\eta\big|_{z=\eta}$, where $\boldsymbol{F}_\mathrm{h}=(F_1,F_2)^{\mathrm{T}}$, and write, according to the Hodge-Weyl decomposition for vector fields in two-dimensional free space,

$$\boldsymbol{v}_\parallel=\nabla\Phi+\nabla^\perp\Psi,$$

where $\Phi=\Delta^{-1}(\nabla\cdot\boldsymbol{v}_\parallel)$, $\Psi=\Delta^{-1}(\nabla^\perp\cdot\boldsymbol{v}_\parallel)=-\Delta^{-1}(\nabla\cdot\boldsymbol{v}_\parallel^\perp)$ and Δ^{-1} is the two-dimensional Newtonian potential. Define a *generalised Dirichlet-Neumann operator* $H(\eta)$ by

$$H(\eta)\Phi=\underline{\mathrm{curl}\,\boldsymbol{A}}\cdot\boldsymbol{n}=\nabla\cdot\boldsymbol{A}_\parallel^\perp,$$

where $(f_1,f_2)^\perp = (f_2,-f_1)$, the underscore denotes evaluation at $z=\eta$ and $\boldsymbol{A}$ is the unique solution of the boundary-value problem

$$\operatorname{curl}\operatorname{curl}\boldsymbol{A} = \alpha\operatorname{curl}\boldsymbol{A} \qquad \text{in } D_\eta, \tag{18}$$

$$\operatorname{div}\boldsymbol{A} = 0 \qquad \text{in } D_\eta, \tag{19}$$

$$\boldsymbol{A}\wedge\boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } z=-1, \tag{20}$$

$$\boldsymbol{A}\cdot\boldsymbol{n} = 0 \qquad \text{at } z=\eta, \tag{21}$$

$$(\operatorname{curl}\boldsymbol{A})_\parallel = \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\boldsymbol{A}_\parallel^\perp) \qquad \text{at } z=\eta. \tag{22}$$

(Note that $\Psi = \Delta^{-1}(\nabla^\perp\cdot(\operatorname{curl}\boldsymbol{A})_\parallel)$ is necessarily given by $\Psi = -\alpha\,\Delta^{-1}(\nabla\cdot\boldsymbol{A}_\parallel^\perp)$ because

$$\Psi = -\Delta^{-1}(\nabla\cdot\operatorname{curl}\boldsymbol{A}_\parallel^\perp) = -\Delta^{-1}(\underline{\operatorname{curl}\operatorname{curl}\boldsymbol{A}}\cdot\boldsymbol{n}) = -\alpha\,\Delta^{-1}(\underline{\operatorname{curl}\boldsymbol{A}}\cdot\boldsymbol{n}) = -\alpha\,\Delta^{-1}(\nabla\cdot\boldsymbol{A}_\parallel^\perp), \tag{23}$$

and that $\boldsymbol{v} = \operatorname{curl}\boldsymbol{A}$ satisfies (7)–(9).)

A straightforward calculation shows that equations (10)–(11) are equivalent to

$$H(\eta)\Phi + \underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n} = 0, \tag{24}$$

$$\frac{1}{2}|\boldsymbol{K}(\eta)\Phi|^2 - \frac{(H(\eta)\Phi + \boldsymbol{K}(\eta)\Phi\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)} + \boldsymbol{K}(\eta)\Phi\cdot\underline{\boldsymbol{u}}_\mathrm{h}^\star + \eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_y = 0, \tag{25}$$

where

$$\boldsymbol{K}(\eta)\Phi := \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(H(\eta)\Phi),$$

and these equations can in fact be reduced to a single equation for the variable η (see Oliveras & Vashal [22] for a simpler version of this equation for irrotational waves). Equation (24) implies that $\Phi = -H(\eta)^{-1}(\underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n})$, whereby (25) yields

$$\mathcal{J}(\eta) = 0, \tag{26}$$

where

$$\mathcal{J}(\eta) := \frac{1}{2}|\boldsymbol{T}(\eta)|^2 - \frac{(-\underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n} + \boldsymbol{T}(\eta)\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)} + \boldsymbol{T}(\eta)\cdot\underline{\boldsymbol{u}}_\mathrm{h}^\star + \eta - \beta\left(\frac{\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_x - \beta\left(\frac{\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}}\right)_y \tag{27}$$

and

$$\boldsymbol{T}(\eta) := -\nabla\left(H(\eta)^{-1}(\underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n})\right) + \alpha\,\nabla^\perp\Delta^{-1}(\underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n}).$$

Equation (26) is invariant under the transformation $\eta(x,y)\mapsto\eta(-x,-y)$ (see the discussion beneath equations (7)–(11)), and in this paper we show that (26) has solutions $\eta_1^\star$, $\eta_2^\star \in H^3(\mathbb{R}^2)$ which satisfy the estimate (17) and are invariant under this transformation.

The operator $\boldsymbol{T}(\eta)$ can also be defined directly in terms of a boundary-value problem. Noting that $\underline{\boldsymbol{u}}^\star\cdot\boldsymbol{n} = \nabla\cdot\boldsymbol{S}(\eta)^\perp$ (and, for later use, that $\underline{\boldsymbol{u}}_\mathrm{h}^\star = \alpha\boldsymbol{c} + \boldsymbol{S}(\eta)$), where

$$\boldsymbol{S}(\eta) = \frac{c_1}{\alpha}\begin{pmatrix}\cos(\alpha\,\eta)-1\\ -\sin(\alpha\,\eta)\end{pmatrix} + \frac{c_2}{\alpha}\begin{pmatrix}\sin(\alpha\,\eta)\\ \cos(\alpha\,\eta)-1\end{pmatrix},$$

we can define

$$\boldsymbol{T}(\eta) := \boldsymbol{M}(\eta)\boldsymbol{S}(\eta),$$

where

$$\boldsymbol{M}(\eta)\boldsymbol{g} := -(\operatorname{curl}\boldsymbol{B})_\parallel,$$

and $\boldsymbol{B}$ solves the boundary-value problem

$$\operatorname{curl}\operatorname{curl}\boldsymbol{B} = \alpha\operatorname{curl}\boldsymbol{B} \qquad \text{in } D_\eta, \tag{28}$$

$$\operatorname{div}\boldsymbol{B} = 0 \qquad \text{in } D_\eta, \tag{29}$$

$$\boldsymbol{B}\wedge\boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } z=-1, \tag{30}$$

$$\boldsymbol{B}\cdot\boldsymbol{n} = 0 \qquad \text{at } z=\eta, \tag{31}$$

$$\nabla\cdot\boldsymbol{B}_{\|}^{\perp} = \nabla\cdot\boldsymbol{g}^{\perp} \qquad \text{at } z=\eta. \tag{32}$$

Any solution to this boundary-value problem satisfies

$$(\operatorname{curl}\boldsymbol{B})_{\|} = \nabla\Phi - \alpha\nabla^{\perp}\Delta^{-1}(\nabla\cdot\boldsymbol{B}_{\|}^{\perp})$$

for some Φ (see equation (23)), so that $\Phi = H(\eta)^{-1}(\nabla\cdot\boldsymbol{g}^{\perp})$ and

$$-(\operatorname{curl}\boldsymbol{B})_{\|} = -\nabla(H(\eta)^{-1}(\nabla\cdot\boldsymbol{g}^{\perp})) + \alpha\nabla^{\perp}\Delta^{-1}(\nabla\cdot\boldsymbol{g}^{\perp}).$$

In Section 2 we show that the solutions to the boundary-value problems (18)–(22) and (28)–(32) depend analytically upon η and use this fact to deduce that the same is true of $H(\eta)$ and $\boldsymbol{M}(\eta)$. We proceed by 'flattening' the fluid domain by means of the transformation $\Sigma\colon D_0 \to D_\eta$ given by

$$\Sigma\colon (\boldsymbol{x}, v) \mapsto (\boldsymbol{x}, v+\sigma(\boldsymbol{x},v)), \qquad \sigma(\boldsymbol{x},v) := \eta(\boldsymbol{x})(1+v),$$

which transforms the boundary-value problems for $\boldsymbol{A}$ and $\boldsymbol{B}$ into equivalent problems for $\tilde{\boldsymbol{A}} := \boldsymbol{A}\circ\Sigma$ and $\tilde{\boldsymbol{B}} := \boldsymbol{B}\circ\Sigma$ in the fixed domain D_0 (equations (36)–(40) and (42)–(46) respectively), and establishing the following results (the function spaces $\mathcal{Z}$, $\dot{H}^s(\mathbb{R}^2)$ and $\check{H}^s(\mathbb{R}^2)$ are defined in Sections 1.5 and 2.1 below).

Theorem 1.4 *There exists a neighbourhood V of the origin in $\mathcal{Z}$ such that*

(i) the boundary-value problem (53)–(57) *has a unique solution $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\eta,\Phi)$ in $H^3(D_0)^3$ which depends analytically upon $\eta \in V$ and $\Phi \in \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)$ (and linearly upon Φ),*

(ii) the boundary-value problem (42)–(46) *has a unique solution $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}(\eta,\boldsymbol{g})$ in $H^3(D_0)^3$ which depends analytically upon $\eta \in V$ and $\boldsymbol{g} \in H^{\frac{5}{2}}(\mathbb{R}^2)^2$ (and linearly upon $\boldsymbol{g}$).*

The analyticity of H and $\boldsymbol{M}$ follows from the above theorem and the facts that

$$H(\eta)(\Phi) = \nabla\cdot\tilde{\boldsymbol{A}}_{\|}^{\perp}, \qquad \boldsymbol{M}(\eta)(\boldsymbol{g}) = -(\operatorname{curl}^{\sigma}\tilde{\boldsymbol{B}})_{\|},$$

where

$$\operatorname{curl}^{\sigma}\tilde{\boldsymbol{B}}(\boldsymbol{x},v) := (\operatorname{curl}\,\boldsymbol{B})\circ\Sigma(\boldsymbol{x},v).$$

Theorem 1.5 *The mappings $\eta \mapsto H(\eta)$ and $\eta \mapsto \boldsymbol{M}(\eta)$ are analytic $V \to L(\dot{H}^{\frac{5}{2}}(\mathbb{R}^2), \check{H}^{\frac{3}{2}}(\mathbb{R}^2))$ and $V \to L(H^{\frac{5}{2}}(\mathbb{R}^2)^2, H^{\frac{3}{2}}(\mathbb{R}^2)^2)$ respectively.*

Our final result follows by noting that $H^3(\mathbb{R}^2)$ is continously embedded in $\mathcal{Z}$, so that

$$U := \{\eta \in H^3(\mathbb{R}^2) : \|\eta\|_{\mathcal{Z}} < M\} \tag{33}$$

is an open neighbourhood of the origin in $H^3(\mathbb{R}^2)$.

Corollary 1.6 *The formula* (27) *defines an analytic function $\mathcal{J} : U \to H^1(\mathbb{R}^2)$ for sufficiently small values of $M > 0$.*

1.4 Reduction

The Ansatz (13) indicates that the Fourier transform of the surface-profile function η for a fully localised solitary wave is concentrated in the region $S = \{(k_1, k_2) : |k_1|, |\frac{k_2}{k_1}| \leq \delta\}$ for some $0 < \delta \ll 1$ (see Figure 3). We therefore decompose $\eta \in H^3(\mathbb{R}^2)$ into the sum of

$$\eta_1 = \chi(\boldsymbol{D})\eta, \qquad \eta_2 = (1 - \chi(\boldsymbol{D}))\,\eta,$$

where χ is the indicator function of the set S, the Fourier transform $\hat{\eta} = \mathcal{F}[\eta]$ of η is defined by

$$\hat{\eta}(k_1, k_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \eta(x, y) \mathrm{e}^{-\mathrm{i}(k_1 x + k_2 y)}\, \mathrm{d}x\, \mathrm{d}y,$$

and $\boldsymbol{D} = (-\mathrm{i}\partial_x, -\mathrm{i}\partial_y)^T$. Setting

$$\boldsymbol{c} = (1 - \varepsilon^2)\boldsymbol{c}_0,$$

choosing β sufficiently large and writing equation (26) as

$$\chi(\boldsymbol{D})\mathcal{J}(\eta_1 + \eta_2) = 0,$$
$$(1 - \chi(\boldsymbol{D}))\,\mathcal{J}(\eta_1 + \eta_2) = 0,$$

we find that the second equation is solvable for η_2 as a function of η_1 for sufficiently small values of ε; the first therefore reduces to

$$\chi(\boldsymbol{D})\mathcal{J}(\eta_1 + \eta_2(\eta_1)) = 0$$

upon inserting $\eta_2 = \eta_2(\eta_1)$. Finally, the scaling

$$\eta_1(x, y) = \varepsilon^2\zeta(X, Y), \qquad X = \varepsilon x,\ Y = \varepsilon^2 y$$

transforms the reduced equation into a perturbation of the equation

$$\varepsilon^{-2} g_\varepsilon(\boldsymbol{D})\zeta + 2\zeta + d_\alpha \chi_\varepsilon(\boldsymbol{D})\zeta^2 = 0, \tag{34}$$

where $g_\varepsilon(k_1, k_2) = g(\varepsilon k_1, \varepsilon^2 k_2)$ and $\chi_\varepsilon(k_1, k_2) = \chi(\varepsilon k_1, \varepsilon^2 k_2)$ (see Sections 3 and 4; the reduced equation is stated precisely in equation (75)). Note that δ is a small, but fixed constant while ε is a small parameter whose maximum value depends upon δ.

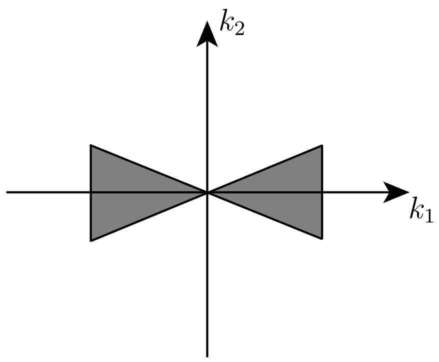

Figure 3: The set $S = \{(k_1, k_2) : |k_1| \leq \delta, \left|\frac{k_2}{k_1}\right| \leq \delta\}$.

Equation (34) is a *full-dispersion* version of the stationary KP-I equation (14) since it retains the linear part of the original equation (26); noting that

$$\varepsilon^{-2} g_\varepsilon(k_1, k_2) = (\beta - \beta_0)k_1^2 + \sec^2 \tfrac{1}{2}\alpha \frac{k_2^2}{k_1^2} + O(\varepsilon),$$

we recover the fully reduced model equation in the formal limit $\varepsilon = 0$. In Section 5 we demonstrate that equation (34) for ζ has solutions ζ_1^ε, ζ_2^ε which satisfy $\zeta_k^\varepsilon \to \pm\zeta_k^\star$ as $\varepsilon \to 0$ in a suitable function space (see Theorem 5.2). The key step is the nondegeneracy result given in Lemma 1.1(iii) which allows one to apply a suitable variant of the implicit-function theorem. For this purpose we exploit the fact that the reduction procedure preserves the invariance of equation (26) under $\eta(x,y) \mapsto \eta(-x,-y)$, so that equation (34) is invariant under $\zeta(x,y) \mapsto \zeta(-x,-y)$; restricting to a space of functions with this invariance, we find that the kernel of the appropriate linearisation is trivial since $\partial_x\zeta_k^\star$, $\partial_y\zeta_k^\star$ do not have this invariance.

The perturbation argument used in Section 5 was developed by Groves [10] in the context of two-dimensional irrotational solitary waves and applied to three-dimensional irrotational fully localised solitary waves on water of infinite depth by Buffoni, Groves & Wahlén [2]. It has also been applied to the Whitham equation by Stefanov & Wright [24] and to a full dispersion KP-I equation (which differs from (34)) by Ehrnström & Groves [9].

1.5 Function spaces

We work with the standard function spaces $H^n(D_\eta)$ for $n \in \mathbb{N}_0$ in the fluid domain together with $L^p(\mathbb{R}^2)$ for $p \geq 1$, $W^{n,\infty}(\mathbb{R}^2)$ for $n \in \mathbb{N}_0$ and

$$H^s(\mathbb{R}^2) = \{\eta \in L^2(\mathbb{R}^2) : (1+|\boldsymbol{k}|^2)^{\frac{1}{2}s}\hat{\eta} \in L^2(\mathbb{R}^2)\}, \qquad \|\eta\|_s^2 = \int_{\mathbb{R}^2} (1+|\boldsymbol{k}|^2)^s |\hat{\eta}(\boldsymbol{k})|^2 \,\mathrm{d}\boldsymbol{k}$$

for $s \geq 0$ in the plane; the definitions are extended componentwise to vector-valued functions. The nonstandard spaces

$$\dot{H}^s(\mathbb{R}^2) = \{\eta \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \nabla\eta \in H^{s-1}(\mathbb{R}^2)^2\}/\mathbb{R}, \quad \|\eta\|_{\dot{H}^s} := \|\nabla\eta\|_{s-1} \cong \|\langle\boldsymbol{k}\rangle^{s-1}|\boldsymbol{k}|\hat{\eta}\|_0, \qquad s \geq 1,$$
$$\check{H}^s(\mathbb{R}^2) = \{\eta \in L^2(\mathbb{R}^2) : \Delta^{-1}\eta \in \dot{H}^{s+2}(\mathbb{R}^2)\}, \quad \|\eta\|_{\check{H}^s} := \|\Delta^{-1}\eta\|_{\dot{H}^{s+2}} \cong \|\langle\boldsymbol{k}\rangle^{s+1}|\boldsymbol{k}|^{-1}\hat{\eta}\|_0, \qquad s \geq 0,$$

where $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ denotes the space of locally square integrable functions in the plane, and the scale $\{Y_s, \|\cdot\|_s\}_{s\geq 0}$, where

$$Y_s = \left\{\eta \in L^2(\mathbb{R}^2) : \left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}s}\hat{\eta} \in L^2(\mathbb{R}^2)\right\}, \quad \|\eta\|_{Y_s} := \left\|\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}s}\hat{\eta}\right\|_0,$$

are also used. Note that $\Delta\colon \dot{H}^{s+2}(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$ is injective so that the definition of $\check{H}^s(\mathbb{R}^2)$ makes sense.

Proposition 1.7

(i) The space Y_1 is continuously embedded in $L^p(\mathbb{R}^2)$ for $2 \leq p \leq 6$.

(ii) The space Y_1 is compactly embedded in $L^2(|\mathbf{x}| < R)$ for each $R > 0$.

(iii) The space Y_s is continuously embedded in $C_\mathrm{b}(\mathbb{R}^2) := C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for $s > \frac{3}{2}$.

Proof. Parts (i) and (ii) are given by respectively Ehrnström & Groves [8, Proposition 2.2(i)] and de Bouard & Saut [7, Lemma 3.3]. Turning to part (iii), note that

$$\|\eta\|_\infty \lesssim \int_{\mathbb{R}^2} |\hat{\eta}(k)|\,\mathrm{d}k = \int_{\mathbb{R}^2}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-\frac{1}{2}s}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}s}|\hat{\eta}(\boldsymbol{k})|\,\mathrm{d}\boldsymbol{k} \leq \|\eta\|_{Y_s} I^{\frac{1}{2}},$$

where

$$I = \int_{\mathbb{R}^2}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-s}\mathrm{d}\boldsymbol{k} = \int_{\mathbb{R}^2}(1+|\boldsymbol{k}|^2)^{-s}|k_1|\,\mathrm{d}\boldsymbol{k} < \infty$$

if and only if $s > \frac{3}{2}$. The continuity of η follows from a standard dominated convergence argument. □

Observe that the spaces $\chi(\boldsymbol{D})H^s(\mathbb{R}^2)$ and $\chi(\boldsymbol{D})Y_s$, $s \geq 0$ of 'truncated' functions all coincide and have equivalent norms. In Sections 3 and 4 we identify in particular $\chi(\boldsymbol{D})H^3(\mathbb{R}^2)$ with $\chi(\boldsymbol{D})Y_1$ and equip it with the scaled norm

$$|||\eta|||^2 := \int_{\mathbb{R}^2}\left(1+\varepsilon^{-2}k_1^2+\varepsilon^{-2}\frac{k_2^2}{k_1^2}\right)|\hat{\eta}(\boldsymbol{k})|^2\,\mathrm{d}\boldsymbol{k} \tag{35}$$

in anticipation of the KP scaling.

Proposition 1.8 *The estimate $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} \lesssim \varepsilon|||\eta_1|||$ holds for each $\eta_1 \in \chi(\boldsymbol{D})Y_1$.*

Proof. Observe that

$$\int_{\mathbb{R}^2} |\hat{\eta}_1(\boldsymbol{k})|\,\mathrm{d}\boldsymbol{k} = \int_{\mathbb{R}^2} \left(1+\varepsilon^{-2}k_1^2+\varepsilon^{-2}\tfrac{k_2^2}{k_1^2}\right)^{-\frac{1}{2}} \left(1+\varepsilon^{-2}k_1^2+\varepsilon^{-2}\tfrac{k_2^2}{k_1^2}\right)^{\frac{1}{2}} |\hat{\eta}_1(\boldsymbol{k})|\,\mathrm{d}\boldsymbol{k} \lesssim |\!|\!|\eta_1|\!|\!| I^{\frac{1}{2}},$$

where

$$I = \int_S \frac{1}{1+\varepsilon^{-2}k_1^2+\varepsilon^{-2}\frac{k_2^2}{k_1^2}}\,\mathrm{d}\boldsymbol{k} = 4\varepsilon^2 \int_0^{\delta/\varepsilon}\int_0^{\delta/\varepsilon} \frac{k_1}{1+k_2^2+k_1^2}\,\mathrm{d}k_2\,\mathrm{d}k_1 \lesssim \varepsilon^2. \qquad \square$$

Corollary 1.9 *The estimate $\|\eta_1\|_{n,\infty} \lesssim \varepsilon|\!|\!|\eta_1|\!|\!|$ holds for each $\eta_1 \in \chi(\boldsymbol{D})Y_1$ and each $n \in \mathbb{N}_0$.*

Proof. The result follows from the calculation $\|\eta_1\|_{n,\infty} \lesssim \||\boldsymbol{k}|^n\hat{\eta}\|_{L^1(\mathbb{R}^2)} \lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)}$ and the previous proposition. $\square$

Finally, we introduce the space $Y_s^\varepsilon = \chi_\varepsilon(\boldsymbol{D})Y_s$ (with norm $\|\cdot\|_{Y_s}$), noting the relationship

$$|\!|\!|\eta|\!|\!|^2 = \varepsilon\|\zeta\|_{Y_1}^2, \qquad \eta(x,y) = \varepsilon^2\zeta(\varepsilon x, \varepsilon^2 y)$$

for $\zeta \in Y_1^\varepsilon$. Observe that Y_s^ε coincides with $\chi_\varepsilon(\boldsymbol{D})H^s(\mathbb{R}^2)$ for $\varepsilon > 0$ and with $\chi(\boldsymbol{D})Y_s$ in the limit $\varepsilon \to 0$.

2 Analyticity

2.1 *The boundary-value problems*

In this section we solve the boundary-value problems (18)–(22) and (28)–(32) and use these results to deduce that $H(\eta)$ and $\boldsymbol{M}(\eta)$ depend analytically upon $\eta \in \mathcal{Z}$, where

$$\mathcal{Z} = \{\eta \in \mathscr{S}'(\mathbb{R}^2) : \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} + \|\eta_2\|_3 < \infty\}$$

and

$$\eta_1 = \chi(\boldsymbol{D})\eta, \qquad \eta_2 = (1-\chi(\boldsymbol{D}))\,\eta$$

(see Theorem 2.5(i) below for a precise statement). We proceed by transforming (18)–(22) and (28)–(32) into equivalent boundary-value problems in the fixed domain D_0 by means of the following 'flattening' transformation. Define $\Sigma\colon D_0 \to D_\eta$ by

$$\Sigma\colon (x,y,v) \mapsto (x,y,v+\sigma(x,y,v)), \qquad \sigma(x,y,v) := \eta(x,y)(1+v),$$

and for $f\colon D_\eta \to \mathbb{R}$ and $\boldsymbol{F}\colon D_\eta \to \mathbb{R}^3$ write $\tilde{f} = f\circ\Sigma$, $\tilde{\boldsymbol{F}} = \boldsymbol{F}\circ\Sigma$ and use the notation

$$\begin{aligned}
\operatorname{grad}^\sigma \tilde{f}(x,y,v) &:= (\operatorname{grad} f)\circ\Sigma(x,y,v),\\
\operatorname{div}^\sigma \tilde{f}(x,y,v) &:= (\operatorname{div} f)\circ\Sigma(x,y,v),\\
\operatorname{curl}^\sigma \tilde{\boldsymbol{F}}(x,y,v) &:= (\operatorname{curl} \boldsymbol{F})\circ\Sigma(x,y,v),\\
\Delta^\sigma \tilde{f}(x,y,v) &:= (\Delta f)\circ\Sigma(x,y,v)
\end{aligned}$$

and more generally

$$\partial_x^\sigma := \partial_x - \frac{\partial_x\sigma}{1+\partial_v\sigma}\partial_v, \quad \partial_y^\sigma := \partial_y - \frac{\partial_y\sigma}{1+\partial_v\sigma}\partial_v, \quad \partial_v^\sigma := \frac{\partial_v}{1+\partial_v\sigma}.$$

Equations (18)–(22) are equivalent to the flattened boundary-value problem

$$\operatorname{curl}^\sigma\operatorname{curl}^\sigma \tilde{\boldsymbol{A}} = \alpha\operatorname{curl}^\sigma \tilde{\boldsymbol{A}} \qquad \text{in } D_0, \tag{36}$$

$$\operatorname{div}^\sigma \boldsymbol{A} = 0 \qquad \text{in } D_0, \tag{37}$$

$$\tilde{\boldsymbol{A}}\wedge\boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v=-1, \tag{38}$$

$$\tilde{\boldsymbol{A}}\cdot\boldsymbol{n} = 0 \qquad \text{at } v=0, \tag{39}$$

$$(\operatorname{curl}^\sigma \tilde{\boldsymbol{A}})_\parallel = \nabla\Phi - \alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\tilde{\boldsymbol{A}}_\parallel^\perp) \tag{40}$$

in terms of which

$$H(\eta)\Phi = \nabla \cdot \tilde{\boldsymbol{A}}_{\parallel}^{\perp}, \tag{41}$$

while equations (28)–(32) are equivalent to the flattened boundary-value problem

$$\operatorname{curl}^\sigma \operatorname{curl}^\sigma \tilde{\boldsymbol{B}} = \alpha \operatorname{curl}^\sigma \tilde{\boldsymbol{B}} \qquad \text{in } D_0, \tag{42}$$

$$\operatorname{div}^\sigma \boldsymbol{B} = 0 \qquad \text{in } D_0, \tag{43}$$

$$\tilde{\boldsymbol{B}} \wedge \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v = -1, \tag{44}$$

$$\tilde{\boldsymbol{B}} \cdot \boldsymbol{n} = 0 \qquad \text{at } v = 0, \tag{45}$$

$$\nabla \cdot \tilde{\boldsymbol{B}}_{\parallel}^{\perp} = \nabla \cdot \boldsymbol{g}^{\perp}, \tag{46}$$

in terms of which

$$\boldsymbol{M}(\eta)\boldsymbol{g} = -(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_{\parallel}; \tag{47}$$

note that the orthogonal gradient part of $(\operatorname{curl}^\sigma \tilde{\boldsymbol{B}})_{\parallel}$ is equal to $-\alpha\, \nabla^\perp \Delta^{-1}(\nabla \cdot \tilde{\boldsymbol{B}}_{\parallel}^{\perp})$ for any solution $\tilde{\boldsymbol{B}} \in H^2(D_0)^3$ of (42)–(46).

It is in fact convenient to replace (18)–(22) with an equivalent boundary-value problem. The following proposition was proved by Groves & Horn [11, Proposition 4.6] (under slightly different regularity assumptions on Φ, η and $\boldsymbol{A}$, the change in which does not affect the proof).

Proposition 2.1 *Suppose that $\Phi \in \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)$ and η lies in a sufficiently small neighbourhood of the origin in $\mathcal{Z}$. A function $\boldsymbol{A} \in H^3(D_\eta)^3$ solves* (18)–(22) *if and only if it satisfies the boundary-value problem*

$$\begin{aligned}
-\Delta \boldsymbol{A} &= \alpha \operatorname{curl} \boldsymbol{A} && \textit{in } D_\eta,\\
\boldsymbol{A} \wedge \boldsymbol{e}_3 &= \boldsymbol{0} && \textit{at } z = -1,\\
\partial_z A_3 &= 0 && \textit{at } z = -1,\\
\boldsymbol{A} \cdot \boldsymbol{n} &= 0 && \textit{at } z = \eta,\\
(\operatorname{curl} \boldsymbol{A})_{\parallel} &= \nabla\Phi - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \boldsymbol{A}_{\parallel}^{\perp}).
\end{aligned}$$

Corollary 2.2 *Suppose that $\Phi \in \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)$ and η lies in a sufficiently small neighbourhood of the origin in $\mathcal{Z}$. A function $\tilde{\boldsymbol{A}} \in H^3(D_0)^3$ solves* (36)–(40) *if and only if it satisfies the boundary-value problem*

$$-\Delta^\sigma \tilde{\boldsymbol{A}} = \alpha \operatorname{curl}^\sigma \tilde{\boldsymbol{A}} \qquad \textit{in } D_0, \tag{48}$$

$$\tilde{\boldsymbol{A}} \wedge \boldsymbol{e}_3 = \boldsymbol{0} \qquad \textit{at } v = -1, \tag{49}$$

$$\partial_v \tilde{A}_3 = 0 \qquad \textit{at } v = -1, \tag{50}$$

$$\tilde{\boldsymbol{A}} \cdot \boldsymbol{n} = 0 \qquad \textit{at } v = 0, \tag{51}$$

$$(\operatorname{curl}^\sigma \tilde{\boldsymbol{A}})_{\parallel} = \nabla\Phi - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \tilde{\boldsymbol{A}}_{\parallel}^{\perp}). \tag{52}$$

We proceed by rewriting (48)–(52) as

$$-\Delta \tilde{\boldsymbol{A}} - \alpha \operatorname{curl} \tilde{\boldsymbol{A}} = \boldsymbol{H}^\sigma(\tilde{\boldsymbol{A}}) \qquad \text{in } D_0, \tag{53}$$

$$\tilde{\boldsymbol{A}} \wedge \boldsymbol{e}_3 = \boldsymbol{0} \qquad \text{at } v = -1, \tag{54}$$

$$\partial_v \tilde{A}_3 = 0 \qquad \text{at } v = -1, \tag{55}$$

$$\tilde{\boldsymbol{A}} \cdot \boldsymbol{e}_3 = g^\sigma(\tilde{\boldsymbol{A}}) \qquad \text{at } v = 0, \tag{56}$$

$$(\underline{\operatorname{curl} \tilde{\boldsymbol{A}}})_{\mathrm{h}} + \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \underline{\tilde{\boldsymbol{A}}}_{\mathrm{h}}^{\perp}) = \boldsymbol{h}^\sigma(\tilde{\boldsymbol{A}}) + \nabla\Phi, \tag{57}$$

where

$$\begin{aligned}
\boldsymbol{H}^\sigma(\tilde{\boldsymbol{A}}) &= \Delta^\sigma \tilde{\boldsymbol{A}} + \alpha \operatorname{curl}^\sigma \tilde{\boldsymbol{A}} - \Delta \tilde{\boldsymbol{A}} - \alpha \operatorname{curl} \tilde{\boldsymbol{A}},\\
g^\sigma(\tilde{\boldsymbol{A}}) &= \nabla\eta \cdot \underline{\tilde{\boldsymbol{A}}}_{\mathrm{h}},\\
\boldsymbol{h}^\sigma(\tilde{\boldsymbol{A}}) &= -(\underline{\operatorname{curl}^\eta \tilde{\boldsymbol{A}}})_{\mathrm{h}} + (\underline{\operatorname{curl} \tilde{\boldsymbol{A}}})_{\mathrm{h}} - \nabla\eta(\underline{\operatorname{curl}^\eta \tilde{\boldsymbol{A}}})_3 - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot (\nabla\eta^\perp \underline{\tilde{A}}_3)).
\end{aligned}$$

(With a slight abuse of notation the underscore now denotes evaluation at $v = 0$). The inhomogeneous linear version of the boundary-value problem (53)–(57) was studied by Groves & Horn [11, Proposition 4.9], who in particular give an explicit formula for the solution.

Lemma 2.3 *Suppose that* $s \geq 2$ *and* $\alpha^\star < \frac{1}{2}\pi$. *The boundary-value problem*

$$\begin{aligned}
&-\Delta \tilde{\boldsymbol{A}} - \alpha \operatorname{curl} \tilde{\boldsymbol{A}} = \boldsymbol{H} && \text{in } D_0,\\
&\tilde{\boldsymbol{A}} \wedge \boldsymbol{e}_3 = \boldsymbol{0} && \text{at } v=-1,\\
&\partial_v \tilde{A}_3 = 0 && \text{at } v=-1,\\
&\tilde{\boldsymbol{A}} \cdot \boldsymbol{e}_3 = g && \text{at } v=0,\\
&(\underline{\operatorname{curl} \tilde{\boldsymbol{A}}})_\mathrm{h} + \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \underline{\tilde{\boldsymbol{A}}}_\mathrm{h}^\perp) = \boldsymbol{h}
\end{aligned}$$

has a unique solution $\tilde{\boldsymbol{A}} \in H^s(D_0)^3$ *for each* $g \in H^{s-\frac{1}{2}}(\mathbb{R}^2)$, $\boldsymbol{H} \in H^{s-2}(D_0)^3$, $\boldsymbol{h} \in H^{s-\frac{3}{2}}(\mathbb{R}^2)^2$ *and* $|\alpha| \in [0, \alpha^\star]$. *The solution operator defines a mapping* $H^{s-\frac{1}{2}}(\mathbb{R}^2) \times H^{s-2}(D_0)^3 \times H^{s-\frac{3}{2}}(\mathbb{R}^2)^2 \to H^s(D_0)^3$ *which is bounded uniformly over* $|\alpha| \in [0, \alpha^\star]$.

Lemma 2.3 can be used in particular to study the boundary-value problems

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} \tilde{\boldsymbol{A}}^0 &= \alpha \operatorname{curl} \tilde{\boldsymbol{A}}^0 & \operatorname{curl}\operatorname{curl} \tilde{\boldsymbol{B}}^0 &= \alpha \operatorname{curl} \tilde{\boldsymbol{B}}^0 && \text{in } D_0, && (58)\\
\operatorname{div} \tilde{\boldsymbol{A}}^0 &= 0 & \operatorname{div} \tilde{\boldsymbol{B}}^0 &= 0 && \text{in } D_0, && (59)\\
\tilde{\boldsymbol{A}}^0 \wedge \boldsymbol{e}_3 &= \boldsymbol{0} & \tilde{\boldsymbol{B}}^0 \wedge \boldsymbol{e}_3 &= \boldsymbol{0} && \text{at } v=-1, && (60)\\
\tilde{\boldsymbol{A}}^0 \cdot \boldsymbol{e}_3 &= 0 & \tilde{\boldsymbol{B}}^0 \cdot \boldsymbol{e}_3 &= 0 && \text{at } v=0, && (61)\\
(\underline{\operatorname{curl} \tilde{\boldsymbol{A}}^0})_\mathrm{h} &= \nabla\Phi - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot (\underline{\tilde{\boldsymbol{A}}}^0)_\mathrm{h}^\perp) & \nabla \cdot (\underline{\tilde{\boldsymbol{B}}}^0)_\mathrm{h}^\perp &= \nabla \cdot \boldsymbol{g}^\perp && && (62)
\end{aligned}$$

for $\Phi \in \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^2)$ and $\boldsymbol{g} \in H^{s-\frac{1}{2}}(\mathbb{R}^2)^2$ with $s \geq 2$. The boundary-value problem for $\tilde{\boldsymbol{A}}^0$ has a unique solution $\tilde{\boldsymbol{A}}^0(\Phi) \in H^s(D_0)^3$, and it follows from

$$H(0)\Phi = \nabla \cdot \underline{\tilde{\boldsymbol{A}}}^0(\Phi)_\mathrm{h}^\perp$$

and the explicit formula for $\tilde{\boldsymbol{A}}^0(\Phi)$ given by Groves & Horn that

$$H(0)\Phi = D^2\,\mathtt{t}(D^2), \qquad \mathtt{t}(\mu) = \begin{cases} \dfrac{\tan(\sqrt{\alpha^2-\mu})}{\sqrt{\alpha^2-\mu}}, & \text{if } \mu < \alpha^2,\\[2ex] \dfrac{\tanh(\sqrt{\mu-\alpha^2})}{\sqrt{\mu-\alpha^2}}, & \text{if } \mu \geq \alpha^2 \end{cases}$$

and

$$\boldsymbol{D} = (D_1, D_2)^T = -\mathrm{i}\nabla, \qquad D = |\boldsymbol{D}|.$$

Note that $H(0) \in L(\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^2), \check{H}^{s-\frac{3}{2}}(\mathbb{R}^2))$ is an isomorphism because

$$\|H(0)^{-1}\Psi\|_{\dot{H}^{s-\frac{1}{2}}} = \left\| \langle \boldsymbol{k}\rangle^{s-\frac{3}{2}} |\boldsymbol{k}| \frac{1}{|\boldsymbol{k}|^2 \mathtt{t}(|\boldsymbol{k}|^2)} \hat{\Psi} \right\|_0 = \left\| \langle \boldsymbol{k}\rangle^{s-\frac{1}{2}} |\boldsymbol{k}|^{-1} \frac{\langle \boldsymbol{k}\rangle^{-1}}{\mathtt{t}(|\boldsymbol{k}|^2)} \hat{\Psi} \right\|_0 \lesssim \|\langle \boldsymbol{k}\rangle^{s-\frac{1}{2}} |\boldsymbol{k}|^{-1} \hat{\Psi}\|_0 = \|\Psi\|_{\check{H}^{s-\frac{3}{2}}},$$

where we have used the fact that $\langle \boldsymbol{k}\rangle^{-1}/\mathtt{t}(|\boldsymbol{k}|^2)$ is bounded.

Observe that $\tilde{\boldsymbol{B}}^0(\boldsymbol{g}) := \tilde{\boldsymbol{A}}^0(\Phi)$ with $\Phi = H(0)^{-1}(\nabla \cdot \boldsymbol{g}^\perp)$ solves the boundary-value problem for $\tilde{\boldsymbol{B}}^0$ because

$$\nabla \cdot \boldsymbol{g}^\perp = H(0)\Phi = \nabla \cdot \underline{\tilde{\boldsymbol{A}}}^0(\Phi)_\mathrm{h}^\perp = \nabla \cdot \underline{\tilde{\boldsymbol{B}}}^0(\boldsymbol{g})_\mathrm{h}^\perp;$$

this solution is unique because any other solution $\tilde{\boldsymbol{B}}^0(\boldsymbol{g})$ is equal to $\tilde{\boldsymbol{A}}^0(\Phi)$ with $\Phi = \Delta^{-1}(\nabla \cdot \underline{\operatorname{curl} \tilde{\boldsymbol{B}}^0}(\boldsymbol{g})_\mathrm{h})$, so that

$$H(0)\Phi = \nabla \cdot \underline{\tilde{\boldsymbol{A}}}^0(\Phi)_\mathrm{h}^\perp = \nabla \cdot \underline{\tilde{\boldsymbol{B}}}^0(\boldsymbol{g})_\mathrm{h}^\perp = \nabla \cdot \boldsymbol{g}^\perp.$$

It now follows from

$$\boldsymbol{M}(0)\boldsymbol{g} = -\underline{\operatorname{curl} \tilde{\boldsymbol{B}}^0}(\boldsymbol{g})_\mathrm{h} = -\underline{\operatorname{curl} \tilde{\boldsymbol{A}}^0}(\Phi)_\mathrm{h} = -\nabla\Phi + \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \boldsymbol{g}^\perp)$$

that $\boldsymbol{M}(0) \in L(H^{s-\frac{1}{2}}(\mathbb{R}^2)^2, H^{s-\frac{3}{2}}(\mathbb{R}^2)^2)$ is given by

$$\boldsymbol{M}(0)\boldsymbol{g} = \frac{1}{D^2}\left(\alpha \boldsymbol{D}^\perp + \boldsymbol{D}\,\mathtt{c}(D^2)\right) \boldsymbol{D} \cdot \boldsymbol{g}^\perp, \qquad \mathtt{c}(\mu) = \begin{cases} \sqrt{\alpha^2-\mu}\,\cot(\sqrt{\alpha^2-\mu}), & \text{if } \mu < \alpha^2,\\ \sqrt{\mu-\alpha^2}\,\coth(\sqrt{\mu-\alpha^2}), & \text{if } \mu \geq \alpha^2. \end{cases}$$

Lemma 2.3 is also the key to solving the boundary-value problem (53)–(57).

Theorem 2.4 *There exists a neighbourhood V of the origin in $\mathcal{Z}$ with the property that the boundary-value problem* (53)–(57) *has a unique solution $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\eta,\Phi)$ in $H^3(D_0)^3$ which depends analytically upon $\eta \in V$ and $\Phi \in \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)$ (and linearly upon Φ).*

Proof. The analyticity of $(\eta,\tilde{\boldsymbol{A}}) \mapsto \mathbf{H}^\sigma(\tilde{\boldsymbol{A}})$ at the origin as a mapping $\mathcal{Z}\times H^3(D_0)^3 \to H^1(D_0)^3$ follows from the explicit expression

$$\begin{aligned}\mathbf{H}^\sigma(\tilde{\boldsymbol{A}}) = &-2\frac{1+v}{1+\eta}(\eta_x\partial_{vx}^2\tilde{\boldsymbol{A}}+\eta_y\partial_{vy}^2\tilde{\boldsymbol{A}}) - \frac{1+v}{1+\eta}\Delta\eta\,\partial_v\tilde{\boldsymbol{A}}\\ &+2\frac{1+v}{(1+\eta)^2}|\nabla\eta|^2\partial_v\tilde{\boldsymbol{A}} + \left(\frac{1+v}{1+\eta}\right)^2|\nabla\eta|^2\partial_v^2\tilde{\boldsymbol{A}} - \frac{\eta^2+2\eta}{(1+\eta)^2}\partial_v^2\tilde{\boldsymbol{A}}\\ &-\alpha\frac{\eta}{1+\eta}(-\partial_v\tilde{A}_2,\partial_v\tilde{A}_1,0)^T - \alpha\frac{1+v}{1+\eta}(\eta_y\partial_v\tilde{A}_3,-\eta_x\partial_v\tilde{A}_3,\eta_x\partial_v\tilde{A}_2-\eta_y\partial_v\tilde{A}_1)^T\end{aligned}$$

by writing

$$\frac{1}{1+\eta}=1-\frac{\eta}{1+\eta},\qquad \frac{1}{(1+\eta)^2}=1-\frac{\eta^2+2\eta}{(1+\eta)^2}$$

and noting that

- the bilinear mappings $(\eta,\tilde{f}) \mapsto \eta_x\tilde{f}$, $(\eta,\tilde{f}) \mapsto \eta_y\tilde{f}$ and $(\eta,\tilde{f}) \mapsto \Delta\eta\,\partial_v\tilde{f}$ are bounded $\mathcal{Z}\times H^1(D_0)\to H^1(D_0)$ and $\mathcal{Z}\times H^2(D_0)\to H^1(D_0)$ because

$$\begin{aligned}\|\eta_x\tilde{f}\|_{H^1(D_0)} &\lesssim (\|\eta_{1x}\|_{1,\infty}+\|\eta_{2x}\|_{1,\infty})\|\tilde{f}\|_{H^1(D_0)} \lesssim (\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)}+\|\eta_2\|_3)\|\tilde{f}\|_{H^1(D_0)},\\ \|\eta_y\tilde{f}\|_{H^1(D_0)} &\lesssim (\|\eta_{1y}\|_{1,\infty}+\|\eta_{2y}\|_{1,\infty})\|\tilde{f}\|_{H^1(D_0)} \lesssim (\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)}+\|\eta_2\|_3))\|\tilde{f}\|_{H^1(D_0)}\end{aligned}$$

  and

$$\begin{aligned}\|\Delta\eta\,\partial_v\tilde{f}\|_{H^1(D_0)} &\lesssim \|\Delta\eta_1\|_{1,\infty}\|\partial_v\tilde{f}\|_{H^1(D_0)} + \|\nabla(\Delta\eta_2)\partial_v\tilde{f}\|_{L^2(D_0)} + \|\Delta\eta_2\nabla(\partial_v\tilde{f})\|_{L^2(D_0)}\\ &\lesssim \|\eta_1\|_{3,\infty}\|\tilde{f}\|_{H^2(D_0)} + \|\eta_2\|_3\|\partial_v\tilde{f}\|_{L^\infty(D_0)} + \|\Delta\eta_2\|_{L^4(\mathbb{R}^2)}\|\partial_v\tilde{f}\|_{W^{1,4}(D_0)}\\ &\lesssim (\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)}+\|\eta_2\|_3)\|\tilde{f}\|_{H^3(D_0)},\end{aligned}$$

- the trilinear mapping $(\eta,\rho,\tilde{f}) \mapsto \nabla\eta\cdot\nabla\rho\,\tilde{f}$ is bounded $\mathcal{Z}^2\times H^1(D_0)\to H^1(D_0)$ because

$$\|\nabla\eta\cdot\nabla\rho\,\tilde{f}\|_{H^1(D_0)} \lesssim (\|\nabla\eta_1\|_{1,\infty}+\|\nabla\eta_2\|_{1,\infty})(\|\nabla\rho_1\|_{1,\infty}+\|\nabla\rho_2\|_{1,\infty})\|\tilde{f}\|_{H^1(D_0)} \lesssim \|\eta\|_{\mathcal{Z}}\|\rho\|_{\mathcal{Z}}\|\tilde{f}\|_{H^1(D_0)},$$

- the mapping $\tilde{f} \mapsto (1+v)\tilde{f}$ belongs to $L(H^1(D_0),H^1(D_0))$,
- a function $f:\mathbb{R}\to\mathbb{R}$ which is analytic at the origin (in particular $f(s)=s(1+s)^{-1}$ and $f(s)=(s^2+2s)(1+s)^{-1}$) induces a mapping $W^{1,\infty}(\mathbb{R}^2)\to W^{1,\infty}(\mathbb{R}^2)$ and hence $\mathcal{Z}\mapsto W^{1,\infty}(\mathbb{R}^2)$ which is analytic at the origin,
- the bilinear mapping $(\rho,\tilde{f}) \mapsto \rho\tilde{f}$ is bounded $W^{1,\infty}(\mathbb{R}^2)\times H^1(D_0)\to H^1(D_0)$.

Similar arguments show that $(\eta,\tilde{\boldsymbol{A}}) \mapsto g^\sigma(\tilde{\boldsymbol{A}})$ and $(\eta,\tilde{\boldsymbol{A}}) \mapsto \boldsymbol{h}^\sigma(\tilde{\boldsymbol{A}})$ are analytic at the origin as mappings $\mathcal{Z}\times H^3(D_0)^3\to H^{\frac{5}{2}}(\mathbb{R}^2)$ and $\mathcal{Z}\times H^3(D_0)^3\to H^{\frac{3}{2}}(\mathbb{R}^2)^2$ respectively.

It follows that the formula

$$\mathcal{H}(\tilde{\boldsymbol{A}},\eta,\Phi) = \begin{pmatrix} -\Delta\tilde{\boldsymbol{A}} - \alpha\,\mathrm{curl}\,\tilde{\boldsymbol{A}} - \boldsymbol{H}^\sigma(\tilde{\boldsymbol{A}})\\ \underline{\tilde{\boldsymbol{A}}}\cdot\boldsymbol{e}_3 - g^\sigma(\tilde{\boldsymbol{A}})\\ (\underline{\mathrm{curl}\,\tilde{\boldsymbol{A}}})_{\mathrm{h}} + \alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\underline{\tilde{\boldsymbol{A}}}_{\mathrm{h}}^\perp) - \boldsymbol{h}^\sigma(\tilde{\boldsymbol{A}}) - \nabla\Phi\end{pmatrix},$$

defines a mapping

$$\mathcal{H}: S\times\mathcal{Z}\times\dot{H}^{\frac{5}{2}}(\mathbb{R}) \to H^1(D_0)^3\times H^{\frac{5}{2}}(\mathbb{R}^2)\times H^{\frac{3}{2}}(\mathbb{R}^2)^2,$$

where $S=\{\tilde{\boldsymbol{A}}\in H^3(D_0)^3 : \tilde{\boldsymbol{A}}\wedge\boldsymbol{e}_3\big|_{v=-1}=\mathbf{0}, \partial_v\tilde{A}_3|_{v=-1}=0\}$, which is analytic at the origin. Furthermore, $\mathcal{H}(\mathbf{0},0,0)=(\mathbf{0},0,\mathbf{0})$, and the calculation

$$\mathrm{d}_1\mathcal{H}[\mathbf{0},0,0](\tilde{\boldsymbol{A}}) = \begin{pmatrix} -\Delta\tilde{\boldsymbol{A}} - \alpha\,\mathrm{curl}\,\tilde{\boldsymbol{A}}\\ \underline{\tilde{\boldsymbol{A}}}\cdot\boldsymbol{e}_3\\ (\underline{\mathrm{curl}\,\tilde{\boldsymbol{A}}})_{\mathrm{h}} + \alpha\nabla^\perp\Delta^{-1}(\nabla\cdot\underline{\tilde{\boldsymbol{A}}}_{\mathrm{h}}^\perp)\end{pmatrix}$$

and Proposition 2.3 show that

$$\mathrm{d}_1\mathcal{H}[\mathbf{0},0,0] : S \to H^1(D_0)^3 \times H^{\frac{5}{2}}(\mathbb{R}^2) \times H^{\frac{3}{2}}(\mathbb{R}^2)^2$$

is an isomorphism. The analytic implicit-function theorem (Buffoni & Toland [3, Theorem 4.5.3]) asserts the existence of open neighbourhoods V_1 and V_2 of the origin in respectively $\mathcal{Z} \times \dot{H}^{\frac{5}{2}}(\mathbb{R})$ and S such that the equation

$$\mathcal{H}(\tilde{\boldsymbol{A}}, \eta, \Phi) = (\mathbf{0}, 0, \mathbf{0})$$

and hence the boundary-value problem (53)–(57) has a unique solution $\tilde{\boldsymbol{A}}_0 = \tilde{\boldsymbol{A}}_0(\eta, \Phi)$ in V_2 for each $(\eta, \Phi) \in V_1$; furthermore $\tilde{\boldsymbol{A}}_0(\eta, \Phi)$ depends analytically upon η and Φ. Since $\tilde{\boldsymbol{A}}_0$ depends linearly upon Φ one can without loss of generality take $V_1 = V \times \dot{H}^{\frac{5}{2}}(\mathbb{R})$, and clearly $V_2 = S$ (with $\Phi = 0$ the construction yields a unique solution in a neighbourhood of the origin in S, which is evidently the zero solution). □

The corresponding result for the boundary-value problem (42)–(46), together with the analyticity of the operators H and M, is now readily deduced.

Theorem 2.5

(i) The mappings $\eta \mapsto H(\eta)$ and $\eta \mapsto \boldsymbol{M}(\eta)$ are analytic $V \to L(\dot{H}^{\frac{5}{2}}(\mathbb{R}^2), \check{H}^{\frac{3}{2}}(\mathbb{R}^2))$ and $V \to L(H^{\frac{5}{2}}(\mathbb{R}^2)^2, H^{\frac{3}{2}}(\mathbb{R}^2)^2)$ respectively.

(ii) The boundary-value problem (42)–(46) *has a unique solution $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}(\eta, \boldsymbol{g})$ in $H^3(D_0)^3$ which depends analytically upon $\eta \in V$ and $\boldsymbol{g} \in H^{\frac{5}{2}}(\mathbb{R}^2)^2$ (and linearly upon $\boldsymbol{g}$).*

Proof. The analyticity of $H(\cdot)\colon V \to L(\dot{H}^{\frac{5}{2}}(\mathbb{R}^2), \check{H}^{\frac{3}{2}}(\mathbb{R}^2))$ follows from Theorem 2.4 and equation (41). Since $H(0) \in L(\dot{H}^{\frac{5}{2}}(\mathbb{R}^2), \check{H}^{\frac{3}{2}}(\mathbb{R}^2))$ is isomorphism we conclude that $H(\eta) \in L(\dot{H}^{\frac{5}{2}}(\mathbb{R}^2), \check{H}^{\frac{3}{2}}(\mathbb{R}^2))$ is an isomorphism for each $\eta \in V$ and that $H(\eta)^{-1} \in L(\check{H}^{\frac{3}{2}}(\mathbb{R}^2), \dot{H}^{\frac{5}{2}}(\mathbb{R}^2))$ also depends analytically upon $\eta \in V$.

The next step is to note that $\tilde{\boldsymbol{B}}(\eta, \boldsymbol{g}) = \tilde{\boldsymbol{A}}(\eta, \Phi)$ with $\Phi = H(\eta)^{-1}(\nabla \cdot \boldsymbol{g}^\perp)$ depends analytically upon η and $\boldsymbol{g}$, and solves (42)–(46) since by construction

$$\nabla \cdot \boldsymbol{g}^\perp = H(\eta)\Phi = \nabla \cdot \tilde{\boldsymbol{A}}(\eta, \Phi)_\parallel^\perp = \nabla \cdot \tilde{\boldsymbol{B}}(\eta, \Phi)_\parallel^\perp.$$

The uniqueness of this solution follows by noting that any other solution $\tilde{\boldsymbol{B}}(\eta, \boldsymbol{g})$ is equal to $\tilde{\boldsymbol{A}}(\eta, \Phi)$ with $\Phi = \Delta^{-1}\nabla \cdot (\mathrm{curl}^\sigma\, \tilde{\boldsymbol{B}})_\parallel$, so that

$$H(\eta)\Phi = \nabla \cdot \tilde{\boldsymbol{A}}(\eta, \Phi)_\parallel^\perp = \nabla \cdot \tilde{\boldsymbol{B}}(\eta, \boldsymbol{g})_\parallel^\perp = \nabla \cdot \boldsymbol{g}^\perp,$$

that is $\Phi = H(\eta)^{-1}(\nabla \cdot \boldsymbol{g}^\perp)$. Finally, the analyticity of $\boldsymbol{M}$ follows from the calculation

$$\begin{aligned} \boldsymbol{M}(\eta)\boldsymbol{g} &= -(\mathrm{curl}^\sigma\, \tilde{\boldsymbol{B}}(\eta, \boldsymbol{g}))_\parallel \\ &= -(\mathrm{curl}^\sigma\, \tilde{\boldsymbol{A}}(\eta, \Phi))_\parallel \\ &= -\nabla\Phi + \alpha\, \nabla^\perp \Delta^{-1}(\nabla \cdot \boldsymbol{g}^\perp) \end{aligned}$$

with $\Phi = H(\eta)^{-1}(\nabla \cdot \boldsymbol{g}^\perp)$. □

We now choose $M > 0$ sufficiently small and note that $H^3(\mathbb{R}^2)$ is continously embedded in $\mathcal{Z}$ and

$$U = \{\eta \in H^3(\mathbb{R}^2) : \|\eta\|_{\mathcal{Z}} < M\}$$

is an open neighbourhood of the origin in $H^3(\mathbb{R}^2)$.

Proposition 2.6 *The mappings $\eta \mapsto \boldsymbol{M}(\eta)$ and $\eta \mapsto \boldsymbol{T}(\eta)$ are analytic are analytic $U \mapsto L(H^{\frac{5}{2}}(\mathbb{R}^2)^2, H^{\frac{3}{2}}(\mathbb{R}^2)^2)$ and $U \to H^{\frac{3}{2}}(\mathbb{R}^2)^2$ respectively.*

Proof. This result follows from Theorem 2.5(i), the formula $\boldsymbol{T}(\eta) = \boldsymbol{M}(\eta)\boldsymbol{S}(\eta)$ and the fact that $\eta \mapsto \boldsymbol{S}(\eta)$ is an analytic mapping $U \to H^3(\mathbb{R}^2)^2$. □

Corollary 2.7 *The formula* (27) *defines an analytic function* $\mathcal{J} : U \to H^1(\mathbb{R}^2)$.

Proof. We proceed by writing the formula as

$$\begin{aligned}\mathcal{J}(\eta) = \frac{1}{2}|\boldsymbol{T}(\eta)|^2 - \frac{1}{2}(-\nabla\cdot\boldsymbol{S}(\eta)^\perp + \boldsymbol{T}(\eta)\cdot\nabla\eta)^2 + \frac{|\nabla\eta|^2(-\nabla\cdot\boldsymbol{S}(\eta)^\perp + \boldsymbol{T}(\eta)\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)} \\ + \boldsymbol{c}\cdot\boldsymbol{T}(\eta) + \alpha\boldsymbol{T}(\eta)\cdot\boldsymbol{S}(\eta) + \eta - \beta\Delta\eta \\ + \beta\left(\frac{|\nabla\eta|^2\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_x + \beta\left(\frac{|\nabla\eta|^2\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_y,\end{aligned}$$

from which the result follows because $\eta \mapsto \boldsymbol{S}(\eta)$, $\eta \mapsto \boldsymbol{T}(\eta)$ and

$$\eta \mapsto \frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}, \qquad \eta \mapsto \frac{|\nabla\eta|^2}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}$$

are analytic mappings $U \to H^3(\mathbb{R}^2)^2$, $U \to H^{\frac{3}{2}}(\mathbb{R}^2)^2$ and $U \to H^2(\mathbb{R}^2)$ respectively and $H^{\frac{3}{2}}(\mathbb{R}^2)$ is a Banach algebra. □

2.2 Taylor expansions

The terms in the Taylor expansions

$$\tilde{\boldsymbol{B}}(\eta) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{B}}^k(\eta), \qquad \tilde{\boldsymbol{B}}^k = \frac{1}{k!}\mathrm{d}^k\tilde{\boldsymbol{B}}[0](\eta^{(k)}), \tag{63}$$

of $\tilde{\boldsymbol{B}}$ and

$$\boldsymbol{M}(\eta) = \sum_{k=0}^{\infty} \boldsymbol{M}_k(\eta), \qquad \boldsymbol{M}_k = \frac{1}{k!}\mathrm{d}^k\boldsymbol{M}[0](\eta^{(k)}), \tag{64}$$

of $\boldsymbol{M} : U \mapsto L(H^3(\mathbb{R}^2)^2, H^{\frac{3}{2}}(\mathbb{R}^2)^2)$ can be determined recursively by substituting them into (42)–(46) and (47). It has already been established that

$$\boldsymbol{M}_0\boldsymbol{g} = -(\underline{\mathrm{curl}\,\tilde{\boldsymbol{B}}}^0)_{\mathrm{h}} = \frac{1}{D^2}\boldsymbol{L}\,\boldsymbol{D}\cdot\boldsymbol{g}^\perp, \qquad \boldsymbol{L} = \alpha\boldsymbol{D}^\perp + \mathsf{c}(D^2)\boldsymbol{D}, \tag{65}$$

where $\tilde{\boldsymbol{B}}^0$ is the unique solution of (58)–(62). Observing that $\boldsymbol{M}_0$ also defines an operator in $L(H^{s-\frac{1}{2}}(\mathbb{R}^2)^2, H^{s-\frac{3}{2}}(\mathbb{R}^2)^2)$ (with $\tilde{\boldsymbol{B}}^0 \in H^s(D_0)^3$) for $s \geq 2$, we can also obtain an explicit expression for $M_1(\eta)\boldsymbol{g}$.

Lemma 2.8 *The formula*

$$\begin{aligned}\boldsymbol{M}_1(\eta)\boldsymbol{g} &= \boldsymbol{M}_0(\eta(\boldsymbol{M}_0\boldsymbol{g})^\perp) - \nabla(\eta\nabla\cdot\boldsymbol{g}^\perp) + \alpha\eta(\boldsymbol{M}_0\boldsymbol{g})^\perp \\ &= -\frac{1}{D^2}\boldsymbol{L}\boldsymbol{D}\cdot\left(\eta\frac{1}{D^2}\boldsymbol{L}\boldsymbol{D}\cdot\boldsymbol{g}^\perp\right) + \boldsymbol{D}(\eta\boldsymbol{D}\cdot\boldsymbol{g}^\perp) + \alpha\eta\frac{1}{D^2}\boldsymbol{L}^\perp\boldsymbol{D}\cdot\boldsymbol{g}^\perp\end{aligned} \tag{66}$$

holds for each $\eta \in H^3(\mathbb{R}^2)$ *and each* $\boldsymbol{g} \in H^3(\mathbb{R}^2)^2$.

Proof. Substituting the expansions (63), (64) into (42)–(46) and (47) and equating constant terms shows that $\tilde{\boldsymbol{B}}^0 \in H^{\frac{7}{2}}(D_0)^3 \subseteq H^3(D_0)^3$ solves the boundary-value problem (58)–(62), while equating terms which are linear in η and making the Ansatz

$$\tilde{\boldsymbol{B}}^1 = (v+1)\eta\partial_v\tilde{\boldsymbol{B}}^0 + \tilde{\boldsymbol{C}}$$

leads to

$$\boldsymbol{M}_1(\eta)\tilde{\boldsymbol{g}} = -(\mathrm{curl}\,\tilde{\boldsymbol{C}})_{\mathrm{h}} - (\mathrm{curl}(v+1)\eta\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}} - \eta(\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}^\perp + \nabla\eta^\perp\partial_v\tilde{B}_3^0 - \nabla\eta\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^\perp\Big|_{v=0},$$

where

$$
\begin{aligned}
\operatorname{curl}\operatorname{curl}\tilde{\boldsymbol{C}} &= \alpha\operatorname{curl}\tilde{\boldsymbol{C}} && \text{in } D_0,\\
\operatorname{div}\tilde{\boldsymbol{C}} &= 0 && \text{in } D_0,\\
\tilde{\boldsymbol{C}}\wedge\boldsymbol{e}_3 &= \boldsymbol{0} && \text{at } v=-1,\\
\tilde{\boldsymbol{C}}\cdot\boldsymbol{e}_3 &= -\eta\partial_v\tilde{B}_3^0+\nabla\eta\cdot\tilde{\boldsymbol{B}}_{\mathrm{h}}^0 && \text{at } v=0,\\
\nabla\cdot\tilde{\boldsymbol{C}}_{\mathrm{h}}^{\perp} &= \nabla\cdot(-\eta(\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}-\nabla\eta B_3^0)^{\perp} && \text{at } v=0.
\end{aligned}
$$

Writing $\tilde{\boldsymbol{C}}=\boldsymbol{C}'+\operatorname{grad}\varphi$, where $\varphi\in H^3(D_0)$ is the unique solution of the boundary-value problem

$$
\begin{aligned}
\Delta\varphi &= 0 && \text{in } D_0,\\
\varphi &= 0 && \text{at } v=-1,\\
\varphi_v &= -\eta\partial_v\tilde{B}_3^0+\nabla\eta\cdot\tilde{\boldsymbol{B}}_{\mathrm{h}}^0 && \text{at } v=0,
\end{aligned}
$$

we find that $\boldsymbol{C}'\in H^2(D_0)$ is the unique solution of the boundary-value problem

$$
\begin{aligned}
\operatorname{curl}\operatorname{curl}\boldsymbol{C}' &= \alpha\operatorname{curl}\boldsymbol{C}' && \text{in } D_0,\\
\operatorname{div}\boldsymbol{C}' &= 0 && \text{in } D_0,\\
\boldsymbol{C}'\wedge\boldsymbol{e}_3 &= \boldsymbol{0} && \text{at } v=-1,\\
\boldsymbol{C}'\cdot\boldsymbol{e}_3 &= 0 && \text{at } v=0,\\
\nabla\cdot(\boldsymbol{C}')_{\mathrm{h}}^{\perp} &= \nabla\cdot(-\eta(\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}-\nabla\eta\tilde{B}_3^0)^{\perp} && \text{at } v=0
\end{aligned}
$$

and

$$
\boldsymbol{M}_1(\eta)\tilde{\boldsymbol{g}} = -(\operatorname{curl}\boldsymbol{C}')_{\mathrm{h}}-(\operatorname{curl}(v+1)\eta\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}-\eta(\partial_v\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}^{\perp}+\nabla\eta^{\perp}\partial_v\tilde{B}_3^0-\nabla\eta\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\Big|_{v=0} \tag{67}
$$

because $\operatorname{curl}\operatorname{grad}\phi=\boldsymbol{0}$ and $\nabla\cdot(\underline{\operatorname{grad}\varphi})_{\mathrm{h}}^{\perp}=0$.

Comparing the boundary-value problem for $\boldsymbol{C}'$ with (58)–(62), we find that

$$
\begin{aligned}
(\underline{\operatorname{curl}\boldsymbol{C}'})_{\mathrm{h}} &= \boldsymbol{M}_0\big(\eta(\underline{\partial_v\tilde{\boldsymbol{B}}}^0)_{\mathrm{h}}+\nabla\eta\underline{\tilde{B}}_3^0\big)\\
&= \boldsymbol{M}_0\big(\eta(\underline{\operatorname{curl}\tilde{\boldsymbol{B}}}^0)_{\mathrm{h}}^{\perp}+\nabla(\eta\underline{\tilde{B}}_3^0)\big)\\
&= -\boldsymbol{M}_0(\eta(\boldsymbol{M}_0\boldsymbol{g})^{\perp})
\end{aligned}
$$

because $\boldsymbol{M}\nabla(\cdot)=\boldsymbol{0}$, and explicit calculations show that

$$
-\nabla\eta\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\Big|_{v=0} = -\nabla(\eta\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp})+\eta\nabla\big(\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\big)\Big|_{v=0} = -\nabla(\eta\nabla\cdot\boldsymbol{g}^{\perp})+\eta\nabla\big(\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\big)\Big|_{v=0}
$$

and

$$
-(\operatorname{curl}(v+1)\eta\tilde{\boldsymbol{B}}_v^0)_{\mathrm{h}}-\eta(\tilde{\boldsymbol{B}}^0)_{\mathrm{h}}^{\perp}+\nabla\eta^{\perp}\tilde{\boldsymbol{B}}_{3v}^0+\eta\nabla\big(\nabla\cdot(\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\big)\Big|_{v=0} = \eta\big(\Delta\tilde{\boldsymbol{B}}_{\mathrm{h}}^0-\nabla\operatorname{div}(\tilde{\boldsymbol{B}}^0)\big)^{\perp}\Big|_{v=0} = \eta(\Delta\tilde{\boldsymbol{B}}_{\mathrm{h}}^0)^{\perp}\Big|_{v=0}.
$$

The result follows by inserting these expressions into (67) and noting that

$$
\Delta\tilde{\boldsymbol{B}}_{\mathrm{h}}^0|_{v=0} = -\alpha(\underline{\operatorname{curl}\tilde{\boldsymbol{B}}}^0)_{\mathrm{h}} = \alpha(\boldsymbol{M}_0\boldsymbol{g}). \qquad \square
$$

Remark 2.9 *This method leads to the loss of two derivatives in the individual terms in the formula for $\boldsymbol{M}_1(\eta)$; the overall validity of the formula arises from subtle cancellations between the terms (see Nicholls and Reitich [21, §2.2] for a discussion of this phenomenon in the context of the classical Dirichlet–Neumann operator).*

Explicit expressions for the first few terms in the Taylor expansion

$$
\boldsymbol{T}(\eta)=\sum_{k=0}^{\infty}\boldsymbol{T}_k(\eta), \qquad \boldsymbol{T}_k(\eta)=\frac{1}{k!}\mathrm{d}\boldsymbol{T}^k[0](\eta^{(k)}),
$$

of $\boldsymbol{T}$ can be computed from the formula $\boldsymbol{T}(\eta) = \boldsymbol{M}(\eta)\boldsymbol{S}(\eta)$ using (65), (66) and the corresponding expansion

$$\boldsymbol{S}(\eta) = \sum_{k=1}^{\infty} \boldsymbol{S}_k(\eta)$$

of $\boldsymbol{S}(\eta)$, where

$$\boldsymbol{S}_k(\eta) := \begin{cases} (-1)^{\frac{k-1}{2}} \dfrac{\alpha^{k-1}\,\eta^k}{k!}\boldsymbol{c}^\perp, & k = 1, 3, 5, \ldots, \\ (-1)^{\frac{k}{2}} \dfrac{\alpha^{k-1}\,\eta^k}{k!}\boldsymbol{c}, & k = 2, 4, 6, \ldots, \end{cases}$$

In particular, we find that

$$\boldsymbol{T}_0 = \boldsymbol{0}, \qquad \boldsymbol{T}_1(\eta) = \boldsymbol{M}_0\boldsymbol{S}_1(\eta), \qquad \boldsymbol{T}_2(\eta) = \boldsymbol{M}_0\boldsymbol{S}_2(\eta) + \boldsymbol{M}_1(\eta)\boldsymbol{S}_1(\eta),$$

such that

$$\begin{aligned} \boldsymbol{T}_1(\eta) &= -\boldsymbol{L}\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{D^2}\eta, \\ \boldsymbol{T}_2(\eta) &= \tfrac{1}{2}\alpha\boldsymbol{L}\frac{\boldsymbol{c}\cdot\boldsymbol{D}^\perp}{D^2}\eta^2 - \alpha\eta\boldsymbol{L}^\perp\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{D^2}\eta + \boldsymbol{L}\frac{\boldsymbol{D}}{D^2}\cdot\left(\eta\boldsymbol{L}\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{D^2}\eta\right) - \boldsymbol{D}(\eta(\boldsymbol{c}\cdot\boldsymbol{D})\eta). \end{aligned}$$

Turning to the Taylor expansion

$$\mathcal{J}(\eta) = \sum_{k=0}^{\infty}\mathcal{J}_k(\eta), \qquad \mathcal{J}_k = \frac{1}{k!}\mathrm{d}^k\mathcal{J}[0](\eta^{(k)}),$$

we conclude that

$$\begin{aligned} \mathcal{J}_1(\eta) &= \boldsymbol{T}_1(\eta)\cdot\boldsymbol{c} + \eta - \beta\Delta\eta \\ &= \left(-\frac{1}{D^2}(\boldsymbol{c}\cdot\boldsymbol{L})(\boldsymbol{c}\cdot\boldsymbol{D}) + 1 + \beta D^2\right)\eta, \\ \mathcal{J}_2(\eta) &= \tfrac{1}{2}|\boldsymbol{T}_1(\eta)|^2 - \tfrac{1}{2}(\boldsymbol{c}\cdot\nabla\eta)^2 + \boldsymbol{T}_2(\eta)\cdot\boldsymbol{c} + \alpha\eta\,\boldsymbol{T}_1(\eta)\cdot\boldsymbol{c}^\perp \\ &= \tfrac{1}{2}\left|\boldsymbol{L}\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{D^2}\eta\right|^2 + \tfrac{1}{2}\alpha\frac{1}{D^2}(\boldsymbol{c}\cdot\boldsymbol{L})(\boldsymbol{c}\cdot\boldsymbol{D}^\perp)\eta^2 + \frac{1}{D^2}(\boldsymbol{c}\cdot\boldsymbol{L})\boldsymbol{D}\cdot\left(\eta\boldsymbol{L}\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{D^2}\eta\right) \\ &\qquad - \tfrac{1}{2}(\boldsymbol{c}\cdot\nabla\eta)^2 + \boldsymbol{c}\cdot\nabla(\eta(\boldsymbol{c}\cdot\nabla\eta)) \end{aligned} \tag{68}$$

and

$$\begin{aligned} \mathcal{J}_{\geq 3}(\eta) = \frac{1}{2}\big(2\boldsymbol{T}_1(\eta) + \boldsymbol{T}_{\geq 2}(\eta)\big)\cdot\boldsymbol{T}_{\geq 2}(\eta) - \frac{(\alpha\boldsymbol{S}(\eta)\cdot\nabla\eta + \boldsymbol{T}(\eta)\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)} - \frac{\boldsymbol{c}\cdot\nabla\eta(\alpha\boldsymbol{S}(\eta)\cdot\nabla\eta + \boldsymbol{T}(\eta)\cdot\nabla\eta)}{1+|\nabla\eta|^2} \\ + \frac{(\boldsymbol{c}\cdot\nabla\eta)^2|\nabla\eta|^2}{2(1+|\nabla\eta|^2)} + \boldsymbol{T}_{\geq 3}(\eta)\cdot\boldsymbol{c} + \alpha\boldsymbol{T}_{\geq 2}(\eta)\cdot\boldsymbol{S}(\eta) + \alpha\boldsymbol{T}_1(\eta)\cdot\boldsymbol{S}_{\geq 2}(\eta) \\ + \beta\left(\frac{|\nabla\eta|^2\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_x + \beta\left(\frac{|\nabla\eta|^2\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_y, \end{aligned}$$

where

$$\mathcal{J}_{\geq 3}(\eta) = \sum_{k=3}^{\infty}\mathcal{J}_k(\eta), \quad \boldsymbol{S}_{\geq 2}(\eta) = \sum_{k=2}^{\infty}\boldsymbol{S}_k(\eta), \quad \boldsymbol{T}_{\geq 2}(\eta) = \sum_{k=2}^{\infty}\boldsymbol{T}_k(\eta), \quad \boldsymbol{T}_{\geq 3}(\eta) = \sum_{k=3}^{\infty}\boldsymbol{T}_k(\eta).$$

3 The reduction procedure

In this section we reduce the equation

$$\mathcal{J}(\eta) = 0$$

with

$$\boldsymbol{c} = (1-\varepsilon^2)\boldsymbol{c}_0$$

into a locally equivalent equation for η_1. Clearly $\eta \in U$ solves this equation if and only if

$$\begin{aligned}\chi(\boldsymbol{D})\mathcal{J}(\eta_1+\eta_2) &= 0,\\ (1-\chi(\boldsymbol{D}))\,\mathcal{J}(\eta_1+\eta_2) &= 0,\end{aligned}$$

where $\eta_1 = \chi(\boldsymbol{D})\eta$, $\eta_2 = (1-\chi(\boldsymbol{D}))\eta$, which equations are given explicitly by

$$\begin{aligned}&g(\boldsymbol{D})\eta_1 + 2\varepsilon^2\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1\\ &\qquad - \varepsilon^4\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1 + \chi(\boldsymbol{D})\big(\mathcal{J}_2(\eta_1+\eta_2)+\mathcal{J}_{\geq 3}(\eta_1+\eta_2)\big) = 0, \qquad (69)\\ &g(\boldsymbol{D})\eta_2 + 2\varepsilon^2\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_2\\ &\qquad - \varepsilon^4\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_2 + (1-\chi(\boldsymbol{D}))\,\big(\mathcal{J}_2(\eta_1+\eta_2)+\mathcal{J}_{\geq 3}(\eta_1+\eta_2))\big) = 0, \qquad (70)\end{aligned}$$

where

$$g(\boldsymbol{k}) = -\frac{1}{|\boldsymbol{k}|^2}(\alpha(\boldsymbol{c}_0\cdot\boldsymbol{k}^\perp)(\boldsymbol{c}_0\cdot\boldsymbol{k}) + \mathsf{c}(|\boldsymbol{k}|^2)(\boldsymbol{c}_0\cdot\boldsymbol{k})^2) + 1 + \beta|\boldsymbol{k}|^2;$$

note that (69), (70) hold in respectively $\chi(\boldsymbol{D})H^1(\mathbb{R}^2)$ and $(1-\chi(\boldsymbol{D}))H^1(\mathbb{R}^2)$. We proceed by solving (70) to determine η_2 as a function of η_1 and inserting $\eta_2 = \eta_2(\eta)$ into (69) to derive a reduced equation for η_1. To this end we write (70) in the form

$$\eta_2 = (1-\chi(\boldsymbol{D}))\,g(\boldsymbol{D})^{-1}\mathcal{A}(\eta_1,\eta_2), \qquad (71)$$

where

$$\mathcal{A}(\eta_1,\eta_2) = -2\varepsilon^2\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_2 + \varepsilon^4\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_2 - (1-\chi(\boldsymbol{D}))\,\big(\mathcal{J}_2(\eta_1+\eta_2)+\mathcal{J}_{\geq 3}(\eta_1+\eta_2))\big). \qquad (72)$$

Proposition 3.1 *The mapping* $(1-\chi(\boldsymbol{D}))\,g(\boldsymbol{D})^{-1}$ *defines a bounded linear operator* $H^1(\mathbb{R}^2) \to H^3(\mathbb{R}^2)$.

Proof. This result follows from the facts that $(0,0)$ is a strict global minimum of $\tilde{g}(k_1, \frac{k_2}{k_1})$ with $\tilde{g}(0,0) = 0$ and that $g(\boldsymbol{k}) \gtrsim |\boldsymbol{k}|^2$ as $|\boldsymbol{k}| \to \infty$. □

The next step is to estimate the nonlinear terms on the right-hand side of equation (71). The requisite estimates for $\mathcal{J}_2(\eta)$ are obtained by examining the explicit formula

$$\mathcal{J}_2(\eta) = m(\eta,\eta) - 2\varepsilon^2 m(\eta,\eta) + \varepsilon^4 m(\eta,\eta),$$

where

$$\begin{aligned}m(v,w) = {}&\tfrac{1}{2}\left(\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v\right)\cdot\left(\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right) + \tfrac{1}{2}\alpha\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})(\boldsymbol{c}_0\cdot\boldsymbol{D}^\perp)vw\\ &+ \frac{1}{2D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})\boldsymbol{D}\cdot\left(v\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right) + \frac{1}{2D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})\boldsymbol{D}\cdot\left(w\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v\right)\\ &+ \tfrac{1}{2}((\boldsymbol{c}_0\cdot\boldsymbol{D})v)((\boldsymbol{c}_0\cdot\boldsymbol{D})w) - \tfrac{1}{2}\boldsymbol{c}_0\cdot\boldsymbol{D}(v(\boldsymbol{c}_0\cdot\boldsymbol{D})w) - \tfrac{1}{2}\boldsymbol{c}_0\cdot\boldsymbol{D}(w(\boldsymbol{c}_0\cdot\boldsymbol{D})v) \qquad (73)\end{aligned}$$

(see equation (68)).

Lemma 3.2 *The estimate* $\|m(v,w)\|_1 \lesssim \|v\|_{\mathcal{Z}}\|w\|_3$ *holds for each* v, $w \in H^3(\mathbb{R}^2)$.

Proof. We estimate each of the terms in the formula for m, observing that

$$\left|\frac{1}{|\boldsymbol{k}|^2}(\alpha\boldsymbol{k}^\perp + \mathsf{c}(|\boldsymbol{k}|^2)\boldsymbol{k})\boldsymbol{c}_0\cdot\boldsymbol{k}\right|,\ \left|\frac{1}{|\boldsymbol{k}|^2}(\alpha\boldsymbol{c}_0\cdot\boldsymbol{k}^\perp + \mathsf{c}(|\boldsymbol{k}|^2)\boldsymbol{c}_0\cdot\boldsymbol{k})\boldsymbol{c}_0\cdot\boldsymbol{k}^\perp\right|,\ \left|\frac{1}{|\boldsymbol{k}|^2}(\alpha\boldsymbol{c}_0\cdot\boldsymbol{k}^\perp + \mathsf{c}(|\boldsymbol{k}|^2)\boldsymbol{c}_0\cdot\boldsymbol{k})\boldsymbol{k}\right| \lesssim \langle\boldsymbol{k}\rangle$$

and that

$$\|f(\boldsymbol{D})\hat{v}_1\|_{n,\infty} \leq \|f(\boldsymbol{k})\langle\boldsymbol{k}\rangle^n\hat{v}_1\|_{L^1(\mathbb{R})} \lesssim \|\hat{v}_1\|_{L^1(\mathbb{R})}$$

for all bounded multipliers f because $\hat{v}_1$ has compact support. We find that

$$\begin{aligned}
\left\|\left(\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v\right)\cdot\left(\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right)\right\|_1 &\lesssim \left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v_1\right\|_{1,\infty}\left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right\|_1 + \left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v_2\right\|_2\left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right\|_2\\
&\leq (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})(\boldsymbol{c}_0\cdot\boldsymbol{D}^\perp)vw\right\|_1 &\lesssim \|vw\|_2\\
&\lesssim (\|v_1\|_{2,\infty} + \|v_2\|_2)\|w\|_2\\
&\leq (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})\boldsymbol{D}\cdot\left(v\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right)\right\|_1 &\lesssim \left\|v\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right\|_2\\
&\lesssim (\|v_1\|_{2,\infty} + \|v_2\|_2)\left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}w\right\|_2\\
&\lesssim (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})\boldsymbol{D}\cdot\left(w\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v\right)\right\|_1 &\lesssim \left\|w\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v\right\|_2\\
&\lesssim \|w\|_2\left(\left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v_1\right\|_{2,\infty} + \left\|\boldsymbol{L}\frac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}v_2\right\|_2\right)\\
&\lesssim (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\|(\boldsymbol{c}_0\cdot\boldsymbol{D})v)((\boldsymbol{c}_0\cdot\boldsymbol{D})w)\|_1 &\lesssim (\|(\boldsymbol{c}_0\cdot\boldsymbol{D})v_1\|_{1,\infty}\|\boldsymbol{c}_0\cdot\boldsymbol{D})w\|_1 + \|(\boldsymbol{c}_0\cdot\boldsymbol{D})v\|_2\|(\boldsymbol{c}_0\cdot\boldsymbol{D})w\|_2\\
&\lesssim (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\|\boldsymbol{c}_0\cdot\boldsymbol{D}(v(\boldsymbol{c}_0\cdot\boldsymbol{D})w)\|_1 &\lesssim \|v(\boldsymbol{c}_0\cdot\boldsymbol{D})w)\|_2\\
&\lesssim (\|v_1\|_{2,\infty} + \|v_2\|_2)\|(\boldsymbol{c}_0\cdot\boldsymbol{D})w\|_2\\
&\lesssim (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3,
\end{aligned}$$

$$\begin{aligned}
\|\boldsymbol{c}_0\cdot\boldsymbol{D}(w(\boldsymbol{c}_0\cdot\boldsymbol{D})v)\|_1 &\lesssim \|w(\boldsymbol{c}_0\cdot\boldsymbol{D})v\|_2\\
&\lesssim \|w\|_2(\|(\boldsymbol{c}_0\cdot\boldsymbol{D})v_1\|_{2,\infty} + \|(\boldsymbol{c}_0\cdot\boldsymbol{D})v_2\|_2)\\
&\lesssim (\|\hat{v}_1\|_{L^1(\mathbb{R}^2)} + \|v_2\|_3)\|w\|_3.
\end{aligned}$$

□

Corollary 3.3 *The estimates*

$$\|\mathcal{J}_2(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}\|\eta\|_3$$

and

$$\|\mathrm{d}\mathcal{J}_2[\eta](u)\|_1 \lesssim \|\eta\|_3\|u\|_{\mathcal{Z}}, \qquad \|\mathrm{d}\mathcal{J}_2[\eta](u)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}\|u\|_3$$

hold for all η, $u \in H^3(\mathbb{R}^2)$.

Lemma 3.4 *The estimates*

$$\|\mathcal{J}_{\geq 3}(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3,$$
$$\|\mathrm{d}\mathcal{J}_{\geq 3}[\eta](u)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|u\|_3 + \|\eta\|_{\mathcal{Z}} \|\eta\|_3 \|u\|_{\mathcal{Z}}$$

hold for each $\eta \in U$ *and* $u \in H^3(\mathbb{R}^2)$.

Proof. Writing

$$\boldsymbol{T}_{\geq 2}(\eta) = \boldsymbol{M}_0 \boldsymbol{S}_{\geq 2}(\eta) + \big(\boldsymbol{M}_1(\eta) + \boldsymbol{M}_{\geq 2}(\eta)\big)\boldsymbol{S}_1(\eta),$$
$$\boldsymbol{T}_{\geq 3}(\eta) = \boldsymbol{M}_0 \boldsymbol{S}_{\geq 3}(\eta) + \boldsymbol{M}_{\geq 2}(\eta)\boldsymbol{S}_1(\eta) + \big(\boldsymbol{M}_1(\eta) + \boldsymbol{M}_{\geq 2}(\eta)\big)\boldsymbol{S}_2(\eta),$$

we find by Theorem 2.5(i) and the fact that $\boldsymbol{M}_0 \in L(H^{\frac{5}{2}}(\mathbb{R}^2)^2, H^{\frac{3}{2}}(\mathbb{R}^2)^2)$ that

$$\begin{aligned}
\|\boldsymbol{T}_{\geq 2}(\eta)\|_{\frac{3}{2}} &\lesssim \left\|\eta \frac{\boldsymbol{S}_{\geq 2}(\eta)}{\eta}\right\|_{\frac{5}{2}} + \|\eta\|_{\mathcal{Z}} \|\boldsymbol{S}_1(\eta)\|_{\frac{5}{2}} \\
&\lesssim (\|\eta_1\|_{3,\infty} + \|\eta_2\|_3) \left\|\frac{\boldsymbol{S}_{\geq 2}(\eta)}{\eta}\right\|_3 + \|\eta\|_{\mathcal{Z}} \|\boldsymbol{S}_1(\eta)\|_3 \\
&\lesssim \|\eta\|_{\mathcal{Z}} \|\eta\|_3, \\
\|\boldsymbol{T}_{\geq 3}(\eta)\|_{\frac{3}{2}} &\lesssim \left\|\eta \frac{\boldsymbol{S}_{\geq 3}(\eta)}{\eta}\right\|_{\frac{5}{2}} + \|\eta\|_{\mathcal{Z}}^2 \|\boldsymbol{S}_1(\eta)\|_{\frac{5}{2}} + \|\eta\|_{\mathcal{Z}} \left\|\eta \frac{S_2(\eta)}{\eta}\right\|_{\frac{5}{2}} \\
&\lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3
\end{aligned}$$

and hence that

$$\begin{aligned}
\|\boldsymbol{T}_1(\eta) \cdot \boldsymbol{T}_{\geq 2}(\eta)\|_1 &\lesssim \|\boldsymbol{M}_0 \boldsymbol{S}_1(\eta_1)\|_{1,\infty} \|\boldsymbol{T}_{\geq 2}(\eta)\|_1 + \|\boldsymbol{M}_0 \boldsymbol{S}_1(\eta_2)\|_{\frac{3}{2}} \|\boldsymbol{T}_{\geq 2}(\eta)\|_{\frac{3}{2}} \\
&\lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} \|\boldsymbol{T}_{\geq 2}(\eta)\|_{\frac{3}{2}} + \|\eta_2\|_{\frac{5}{2}} \|\boldsymbol{T}_{\geq 2}(\eta)\|_{\frac{3}{2}} \\
&\lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3, \\
\|\boldsymbol{T}_{\geq 2}(\eta) \cdot \boldsymbol{T}_{\geq 2}(\eta)\|_1 &\lesssim \|\boldsymbol{T}_{\geq 2}(\eta)\|_{\frac{3}{2}}^2 \\
&\lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3^2, \\
\|\boldsymbol{T}_{\geq 2}(\eta) \cdot \boldsymbol{S}(\eta)\|_1 &\lesssim \|\boldsymbol{T}_{\geq 2}(\eta)\|_1 \|\boldsymbol{S}(\eta)\|_{1,\infty} \\
&\lesssim \|\eta\|_{\mathcal{Z}} \|\eta\|_3 (\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty}) \\
&\lesssim \|\eta\|_{\mathcal{Z}} \|\eta\|_3 (\|\eta_1\|_{1,\infty} + \|\eta_2\|_3) \\
&\lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3, \\
\|\boldsymbol{T}_1(\eta) \cdot \boldsymbol{S}_{\geq 2}(\eta)\|_1 &\lesssim \|\boldsymbol{T}_1(\eta)\|_1 \|\boldsymbol{S}_{\geq 2}(\eta)\|_{1,\infty} \\
&\lesssim \|\eta\|_3 (\|\eta_1\|_{1,\infty} + \|\eta_2\|_{1,\infty})^2 \\
&\lesssim \|\eta\|_3 \|\eta\|_{\mathcal{Z}}^2.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\left\|\frac{(\alpha \boldsymbol{S}(\eta) \cdot \nabla\eta + \boldsymbol{T}(\eta) \cdot \nabla\eta)^2}{2(1+|\nabla\eta|^2)}\right\|_1 &\lesssim (\|\boldsymbol{S}(\eta)\|_1 + \|\boldsymbol{T}(\eta)\|_1)^2 \left\|\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}\right\|_{1,\infty} \\
&\lesssim (\|\boldsymbol{S}(\eta)\|_1 + \|\boldsymbol{T}(\eta)\|_1)^2 \|\nabla\eta\|_{1,\infty}^2 \\
&\lesssim \|\eta\|_3^2 (\|\nabla\eta_1\|_{1,\infty} + \|\nabla\eta\|_2)^2 \\
&\lesssim \|\eta\|_3^2 \|\eta\|_{\mathcal{Z}}^2,
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{\boldsymbol{c} \cdot \nabla\eta(\alpha \boldsymbol{S}(\eta) \cdot \nabla\eta + \boldsymbol{T}(\eta) \cdot \nabla\eta)}{1+|\nabla\eta|^2}\right\|_1 &\lesssim (\|\boldsymbol{S}(\eta)\|_1 + \|\boldsymbol{T}(\eta)\|_1) \left\|\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}\right\|_{1,\infty} \\
&\lesssim \|\eta\|_3 \|\eta\|_{\mathcal{Z}}^2
\end{aligned}$$

because $\boldsymbol{S}, \boldsymbol{T}$ are analytic $U \to H^1(\mathbb{R}^2)^2$. Finally, we note that

$$\left(\frac{|\nabla\eta|^2\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_x + \left(\frac{|\nabla\eta|^2\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_y$$
$$= f_1(\eta_x,\eta_y)\eta_{xx} + f_2(\eta_x,\eta_y)\eta_{xy} + f_3(\eta_x,\eta_y)\eta_{yy},$$

where f_1, f_2, f_3 are analytic functions with zeros of order two at the origin. Estimating

$$\|f_1(\eta_x,\eta_y)\eta_{xx}\|_0 \lesssim \|\nabla\eta\|_\infty^2 \|\eta_{xx}\|_0 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3,$$

$$\begin{aligned}\|\nabla(f_1(\eta_x,\eta_y)\eta_{xx})\|_0 &\leq \|\partial_1 f_1(\eta_x,\eta_y)\eta_{xx}\nabla\eta_x + \partial_2 f_1(\eta_x,\eta_y)\eta_{xx}\nabla\eta_y\|_0 + \|f_1(\eta_x,\eta_y)\nabla\eta_{xx})\|_0 \\ &\lesssim \|\nabla\eta\|_\infty(\|\nabla\eta_x\|_{L^4(\mathbb{R}^2)} + \|\nabla\eta_y\|_{L^4(\mathbb{R}^2)})\|\eta_{2xx}\|_{L^4(\mathbb{R}^2)} \\ &\qquad + \|\nabla\eta\|_\infty(\|\nabla\eta_x\|_0 + \|\nabla\eta_y\|_0)\|\eta_{1xx}\|_\infty + \|\nabla\eta\|_\infty^2\|\nabla\eta_{xx}\|_0 \\ &\lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_3,\end{aligned}$$

in which the last fine follows by the continuous embedding $H^1(\mathbb{R}^2) \subseteq L^4(\mathbb{R}^2)$, and $f_2(\eta_x,\eta_y)\eta_{xy}$, $f_3(\eta_x,\eta_y)\eta_{yy}$ similarly, we conclude that

$$\left\|\left(\frac{|\nabla\eta|^2\eta_x}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_x + \left(\frac{|\nabla\eta|^2\eta_y}{(1+|\nabla\eta|^2)^{\frac{1}{2}}(1+(1+|\nabla\eta|^2)^{\frac{1}{2}})}\right)_y\right\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_3.$$

The estimates for the derivatives are obtained in a similar fashion. □

We proceed by solving (71) for η_2 as a function of η_1 using the following following fixed-point theorem, which is proved by a straightforward application of the contraction mapping principle.

Theorem 3.5 *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be Banach spaces, X_1, X_2 be closed, convex sets in, respectively, $\mathcal{X}_1$, $\mathcal{X}_2$ containing the origin and $\mathcal{G}\colon X_1 \times X_2 \to \mathcal{X}_2$ be a smooth mapping. Suppose there exists a continuous mapping $r\colon X_1 \to [0,\infty)$ such that*

$$\|\mathcal{G}(x_1,0)\| \leq \tfrac{1}{2}r, \quad \|\mathrm{d}_2\mathcal{G}[x_1,x_2]\| \leq \tfrac{1}{3}$$

for each $x_2 \in \overline{B}_r(0) \subseteq X_2$ and each $x_1 \in X_1$.

Under these hypotheses there exists for each $x_1 \in X_1$ a unique solution $x_2 = x_2(x_1)$ of the fixed-point equation $x_2 = \mathcal{G}(x_1,x_2)$ satisfying $x_2(x_1) \in \overline{B}_r(0)$. Moreover $x_2(x_1)$ is a smooth function of $x_1 \in X_1$ with

$$\|\mathrm{d}x_2[x_1]\| \leq 2\|\mathrm{d}_1\mathcal{G}[x_1,x_2(x_1)]\|.$$

We apply Theorem 3.5 to equation (71) with

$$\mathcal{X}_1 = \chi(\boldsymbol{D})H^3(\mathbb{R}^2), \qquad \mathcal{X}_2 = (1-\chi(\boldsymbol{D}))\,H^3(\mathbb{R}^2),$$

equipping $\mathcal{X}_1$ with the scaled norm $|||\cdot|||$ defined in (35) and $\mathcal{X}_2$ with the usual norm for $H^3(\mathbb{R}^2)$, and taking

$$X_1 = \{\eta_1 \in \mathcal{X}_1 \colon |||\eta_1||| \leq R_1\}, \qquad X_2 = \{\eta_2 \in \mathcal{X}_2 \colon \|\eta_2\|_3 \leq R_2\};$$

the function $\mathcal{G}$ is given by the right-hand side of (71). Recall that $\mathcal{J}$ is an analytic function $U \to H^1(\mathbb{R}^2)$ (see equation (33)). Using Proposition 1.8 we can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < \frac{1}{2}M$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_2 is then constrained by the requirement that $\|\eta_2\|_3 < \frac{1}{2}M$ for all $\eta_2 \in X_2$.

We proceed by estimating each term appearing in the formula (72) for $\mathcal{A}$ using Corollary 3.3, Lemma 3.4, together with Proposition 1.8 and the estimates

$$\|\eta\|_{\mathcal{Z}} \lesssim \varepsilon|||\eta_1||| + \|\eta_2\|_3, \qquad \|\eta\|_3 \lesssim |||\eta_1||| + \|\eta_2\|_3$$

for $\eta \in H^3(\mathbb{R}^2)$.

Lemma 3.6 *The estimates*

(i) $\|\mathcal{A}(\eta_1,\eta_2)\|_1 \lesssim \varepsilon|\!|\!|\eta_1|\!|\!|^2 + \varepsilon|\!|\!|\eta_1|\!|\!|\|\eta_2\|_3 + |\!|\!|\eta_1|\!|\!|\|\eta_2\|_3^2 + \|\eta_2\|_3^2 + \varepsilon^2\|\eta_2\|_3$,

(ii) $\|\mathrm{d}_1\mathcal{A}[\eta_1,\eta_2]\|_{L(\mathcal{X}_1,H^1(\mathbb{R}^2))} \lesssim \varepsilon|\!|\!|\eta_1|\!|\!| + \varepsilon\|\eta_2\|_3 + \|\eta_2\|_3^2$,

(iii) $\|\mathrm{d}_2\mathcal{A}[\eta_1,\eta_2]\|_{L(\mathcal{X}_2,H^1(\mathbb{R}^2))} \lesssim \varepsilon|\!|\!|\eta_1|\!|\!| + |\!|\!|\eta_1|\!|\!|\|\eta_2\|_3 + \|\eta_2\|_3 + \varepsilon^2$

hold for each $\eta_1 \in X_1$ *and* $\eta_2 \in X_2$.

Theorem 3.7 *Equation* (71) *has a unique solution* $\eta_2 \in X_2$ *which depends smoothly upon* $\eta_1 \in X_1$ *and satisfies the estimates*

$$\|\eta_2(\eta_1)\|_3 \lesssim \varepsilon|\!|\!|\eta_1|\!|\!|^2, \quad \|\mathrm{d}\eta_2[\eta_1]\|_{L(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon|\!|\!|\eta_1|\!|\!|.$$

Proof. Choosing R_2 and ε sufficiently small, one finds $r > 0$ such that $\|\mathcal{G}(\eta_1,0)\|_3 \leq \frac{1}{2}r$ and $\|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_3]\|_{L(\mathcal{X}_2,\mathcal{X}_2)} \leq \frac{1}{3}$ for $\eta_1 \in X_1$, $\eta_2 \in X_2$, and Theorem 3.5 asserts that equation (71) has a unique solution $\eta_2 \in X_2$ which depends smoothly upon $\eta_1 \in X_1$. More precise estimates are obtained by choosing $C > 0$ so that $\|\mathcal{G}(\eta_1,0)\|_3 \leq C\varepsilon|\!|\!|\eta_1|\!|\!|^2$ for $\eta_1 \in X_1$ and writing $r(\eta_1) = 2C\varepsilon|\!|\!|\eta_1|\!|\!|^2$, so that

$$\|\mathrm{d}_1\mathcal{G}[\eta_1,\eta_2]\|_{L(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon|\!|\!|\eta_1|\!|\!|, \quad \|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_2]\|_{L(\mathcal{X}_2,\mathcal{X}_2)} \lesssim 1$$

for $\eta_1 \in X_1$, $\eta_2 \in \overline{B}_{r(\eta_1)}(0) \subseteq X_2$, and the stated estimates for $\eta_2(\eta_1)$ follow from Theorem 3.5. □

Inserting $\eta_2 = \eta_2(\eta_1)$ into (69) yields the reduced equation

$$g(\boldsymbol{D})\eta_1 + 2\varepsilon^2\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1 - \varepsilon^4\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1 + \chi(\boldsymbol{D})\big(\mathcal{J}_2(\eta_1+\eta_2(\eta_1)) + \mathcal{J}_{\geq 3}(\eta_1+\eta_2(\eta_1))\big) = 0 \tag{74}$$

for η_1, which holds in $\chi(\boldsymbol{D})H^1(\mathbb{R}^2)$. This equation is invariant under the reflection $\eta_1(x,y) \mapsto \eta_1(-x,-y)$; a familiar argument shows that it is inherited from the corresponding invariance $\eta_1(x,y) \mapsto \eta_1(-x,-y)$, $\eta_2(x,y) \mapsto \eta_2(-x,-y)$, of (69), (70) when applying Theorem 3.5.

4 Derivation of the reduced equation

In this section we compute the leading-order terms in the reduced equation (74). The first step is to write

$$\begin{aligned}\mathcal{J}_2(\eta_1+\eta_2(\eta_1)) = {} & m(\eta_1,\eta_1) - 2\varepsilon^2 m(\eta_1,\eta_1) + \varepsilon^4 m(\eta_1,\eta_1)\\ & + m(\eta_1,\eta_2(\eta_1)) - 2\varepsilon^2 m(\eta_1,\eta_2(\eta_1)) + \varepsilon^4 m(\eta_1,\eta_2(\eta_1))\\ & + m(\eta_2(\eta_1),\eta_2(\eta_1)) - 2\varepsilon^2 m(\eta_2(\eta_1),\eta_2(\eta_1)) + \varepsilon^4 m(\eta_2(\eta_1),\eta_2(\eta_1))\end{aligned}$$

and examine each of the terms on the right-hand side of this expression individually. The first term is handled by approximating the Fourier-multiplier operators by constants according to Lemma 4.1 below. The order-of-magnitude estimates in this section are computed with respect to the $L^2(\mathbb{R}^2)$-norm, which is equivalent to the $H^s(\mathbb{R}^2)$-norm on the space $\chi(\boldsymbol{D})H^s(\mathbb{R}^2)$ for any $s \geq 0$.

Lemma 4.1 *The estimates*

(i) $\eta_{1x} = O(\varepsilon|\!|\!|\eta_1|\!|\!|)$,

(ii) $\eta_{1z} = O(\varepsilon|\!|\!|\eta_1|\!|\!|)$,

(iii) $\boldsymbol{L}\dfrac{\boldsymbol{c}_0\cdot\boldsymbol{D}}{D^2}\eta_1 = \begin{pmatrix}\alpha\cot\alpha c_{0,1}\\ -\alpha c_{0,1}\end{pmatrix}\eta_1 + O(\varepsilon|\!|\!|\eta_1|\!|\!|)$,

(iv) $\dfrac{1}{D^2}(\boldsymbol{c}_0 \cdot \boldsymbol{L})(\boldsymbol{c}_0 \cdot \boldsymbol{D})\eta_1 = \underbrace{\alpha c_{0,1}(-c_{0,2} + c_{0,1} \cot \alpha)}_{=1} \eta_1 + O(\varepsilon |\!|\!| \eta_1 |\!|\!|)$,

(v) $\dfrac{1}{D^2}(\boldsymbol{c}_0 \cdot \boldsymbol{L})(\boldsymbol{c}_0 \cdot \boldsymbol{D}^\perp)\eta_1\rho_1 = \alpha c_{0,2}(c_{0,2} - c_{0,1} \cot \alpha)\eta_1\rho_1 + O(\varepsilon^2 |\!|\!| \eta_1 |\!|\!| |\!|\!| \rho_1 |\!|\!|) + B_1(\boldsymbol{D})\eta_1\rho_1$,

(vi) $\dfrac{1}{D^2}(\boldsymbol{c}_0 \cdot \boldsymbol{L})\boldsymbol{D} \cdot (\eta_1\rho_1\boldsymbol{w}) = \alpha(-c_{0,2} + c_{0,1} \cot \alpha)\eta_1\rho_1 w_1 + O(\varepsilon^2 |\!|\!| \eta_1 |\!|\!| |\!|\!| \rho_1 |\!|\!|) + B_2(\boldsymbol{D})\eta_1\rho_1$,

hold for all $\eta_1, \rho_1 \in \mathcal{X}_1$, *where* $\boldsymbol{w}$ *is a vector-valued constant,*

$$|B_j(\boldsymbol{k})| \lesssim |\tfrac{k_2}{k_1}|(1 + \tfrac{k_2^2}{k_1^2})^{-1}$$

and $c_{0,1} = c_0 \cos \frac{1}{2}\alpha$, $c_{0,2} = -c_0 \sin \frac{1}{2}\alpha$.

Proof. Parts (i)–(iv) follow from the calculations

$$\begin{aligned}
\|\eta_{1x}\|_0^2 &= \|k_1 \hat{\eta}_1\|_0^2 \\
&\leq \varepsilon^2 |\!|\!| \eta_1 |\!|\!|^2,
\end{aligned}$$

$$\begin{aligned}
\|\eta_{1z}\|_0^2 &= \|k_2 \hat{\eta}_1\|_0^2 \\
&= \|k_1 \tfrac{k_2}{k_1} \hat{\eta}_1\|_0^2 \\
&\leq \delta^2 \|k_1 \hat{\eta}_1\|_0^2 \\
&\lesssim \varepsilon^2 |\!|\!| \eta_1 |\!|\!|^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \boldsymbol{L} \frac{\boldsymbol{c}_0 \cdot \boldsymbol{D}}{D^2} \eta_1 - \begin{pmatrix} \alpha \cot \alpha \, c_{0,1} \\ -\alpha \, c_{0,1} \end{pmatrix} \eta_1 \right\|_0^2 \\
&\quad = \Bigg\| \underbrace{\left(\frac{1}{1 + \frac{k_2^2}{k_1^2}} \left(\alpha \tfrac{k_2}{k_1} + \mathsf{c}(|\boldsymbol{k}|^2) \right) (c_{0,1} + c_{0,2} \tfrac{k_2}{k_1}) - \alpha \cot \alpha \, c_{0,1} \right)}_{= O(|(k_1, \frac{k_2}{k_1}|)} \hat{\eta}_1 \Bigg\|_0^2 \\
&\qquad + \Bigg\| \underbrace{\left(\frac{1}{1 + \frac{k_2^2}{k_1^2}} \left(-\alpha + \mathsf{c}(|\boldsymbol{k}|^2) \tfrac{k_2}{k_1} \right) (c_{0,1} + c_{0,2} \tfrac{k_2}{k_1}) + \alpha c_{0,1} \right)}_{= O(|(k_1, \frac{k_2}{k_1}|)} \hat{\eta}_1 \Bigg\|_0^2 \\
&\quad \lesssim \|k_1 \hat{\eta}_1\|_0^2 + \|\tfrac{k_2}{k_1} \hat{\eta}_1\|_0^2 \\
&\quad \lesssim \varepsilon^2 |\!|\!| \eta_1 |\!|\!|^2,
\end{aligned}$$

$$\begin{aligned}
&\left\| \frac{1}{D^2}(\boldsymbol{c}_0 \cdot \boldsymbol{L})(\boldsymbol{c}_0 \cdot \boldsymbol{D})\eta_1 - \alpha c_{0,1}(-c_{0,2} + c_{0,1} \cot \alpha)\eta_1 \right\|_0^2 \\
&\quad = \Bigg\| \underbrace{\left(\frac{1}{1 + \frac{k_2^2}{k_1^2}} \left(\alpha(c_{0,1} \tfrac{k_2}{k_1} - c_{0,2}) + \mathsf{c}(|\boldsymbol{k}|^2)(c_{0,1} + c_{0,2} \tfrac{k_2}{k_1}) \right) (c_{0,1} + c_{0,2} \tfrac{k_2}{k_1}) - \alpha c_{0,1}(-c_{0,2} + c_{0,1} \cot \alpha) \right)}_{= O(|(k_1, \frac{k_2}{k_1}|)} \hat{\eta}_1 \Bigg\|_0^2 \\
&\quad \lesssim \|k_1 \hat{\eta}_1\|_0^2 + \|\tfrac{k_2}{k_1} \hat{\eta}_1\|_0^2 \\
&\quad \lesssim \varepsilon^2 |\!|\!| \eta_1 |\!|\!|^2.
\end{aligned}$$

Turning to parts (v) and (vi), note that

$$\begin{aligned}&\left\|\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})(\boldsymbol{c}_0\cdot\boldsymbol{D}^\perp)\eta_1\rho_1-\alpha c_{0,2}(c_{0,2}-c_{0,1}\cot\alpha)\eta_1\rho_1\right\|_0^2\\&=\Bigg\|\underbrace{\left(\frac{1}{1+\frac{k_2^2}{k_1^2}}\left(\alpha(c_{0,1}\tfrac{k_2}{k_1}-c_{0,2})+\mathsf{c}(|\boldsymbol{k}|^2)(c_{0,1}+c_{0,2}\tfrac{k_2}{k_1})\right)\left(c_{0,1}\tfrac{k_2}{k_1}-c_{0,2}\right)-\alpha c_{0,2}(c_{0,2}-c_{0,1}\cot\alpha)\right)}_{=O(|(k_1,\frac{k_2}{k_1})|(1+\frac{k_2^2}{k_1^2})^{-1})}\mathcal{F}[\eta_1\rho_1]\Bigg\|_0^2,\\&\left\|\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{L})\boldsymbol{D}\cdot(\eta_1\rho_1\boldsymbol{w})-\alpha(-c_{0,2}+c_{0,1}\cot\alpha)\eta_1\rho_1w_1\right\|_0^2\\&=\Bigg\|\underbrace{\left(\frac{1}{1+\frac{k_2^2}{k_1^2}}\left(\alpha(c_{0,1}\tfrac{k_2}{k_1}-c_{0,2})+\mathsf{c}(|\boldsymbol{k}|^2)(c_{0,1}+c_{0,2}\tfrac{k_2}{k_1})\right)-\alpha(-c_{0,2}+c_{0,1}\cot\alpha)\right)}_{=O(|(k_1,\frac{k_2}{k_1})|(1+\frac{k_2^2}{k_1^2})^{-1})}\mathcal{F}[\eta_1\rho_1]w_1\Bigg\|_0^2\\&\qquad+\Bigg\|\underbrace{\left(\frac{1}{1+\frac{k_2^2}{k_1^2}}\left(\alpha(c_{0,1}\tfrac{k_2}{k_1}-c_{0,2})+\mathsf{c}(|\boldsymbol{k}|^2)(c_{0,1}+c_{0,2}\tfrac{k_2}{k_1})\right)\tfrac{k_2}{k_1}\right)}_{=O(|(k_1,\frac{k_2}{k_1})|(1+\frac{k_2^2}{k_1^2})^{-1})}\mathcal{F}[\eta_1\rho_1]w_2\Bigg\|_0^2\end{aligned}$$

and

$$\left\|\frac{|k_1|}{1+\frac{k_2^2}{k_1^2}}\mathcal{F}[\eta_1\rho_1]\right\|_0\leq\|k_1\mathcal{F}[\eta_1\rho_1]\|_0=\|(\eta_1\rho_1)_x\|_0\leq\|k_1\hat{\eta}_1\|_0\|\rho_1\|_\infty+\|\eta_1\|_\infty\|k_1\hat{\rho}_1\|_0\leq\varepsilon^2|||\eta_1|||\,|||\rho_1|||.\quad\square$$

Corollary 4.2 *The estimate*

$$m(\eta_1,\rho_1)=d_\alpha\eta_1\rho_1+B(\boldsymbol{D})\eta_1\rho_1+O(\varepsilon^2|||\eta_1|||\,|||\rho_1|||),$$

where

$$|B(\boldsymbol{k})|\lesssim|\tfrac{k_2}{k_1}|(1+\tfrac{k_2^2}{k_1^2})^{-1}$$

and $d_\alpha=\alpha\operatorname{cosec}\alpha+\frac{1}{2}\alpha\cot\alpha$, *holds for all* η_1, $\rho_1\in\mathcal{X}_1$.

Proof. This result is obtained by estimating each of the terms in the formula (68) for $\mathcal{J}_2$ using Lemma 4.1. □

The remaining terms in the reduced equation are treated in the next lemma, which follows directly from Lemmata† 3.2, 3.4, Theorem 3.7 and Corollary 4.2.

Lemma 4.3 *The estimates*

$$\varepsilon^2m(\eta_1,\eta_1)=\underline{O}(\varepsilon^2|||\eta_1|||^2),\qquad m(\eta_1,\eta_2(\eta_1))=\underline{O}(\varepsilon^2|||\eta_1|||^3),\qquad m(\eta_2(\eta_1),\eta_2(\eta_1))=\underline{O}(\varepsilon^2|||\eta_1|||^4).$$

and

$$\mathcal{J}_{\geq3}(\eta_1+\eta_2(\eta_1))=\underline{O}(\varepsilon^2|||\eta_1|||^3)$$

hold for all $\eta_1\in X_1$. *Here the symbol* $\underline{O}(\varepsilon^\gamma|||\eta_1|||^r)$ *(with* $\gamma\geq0$, $r\geq1$*) denotes a smooth function* $\mathcal{R}^\varepsilon:X_1\to H^1(\mathbb{R}^2)$ *which satisfies the estimates*

$$\|\mathcal{R}^\varepsilon(\eta_1)\|_1\lesssim\varepsilon^\gamma|||\eta_1|||^r,\quad\|\mathrm{d}\mathcal{R}^\varepsilon[\eta_1]\|_{L(\mathcal{X}_1,H^1(\mathbb{R}^2))}\lesssim\varepsilon^\gamma|||\eta_1|||^{r-1}$$

for each $\eta_1\in X_1$.

Altogether we conclude that (74) can be written as

$$g(\boldsymbol{D})\eta_1+2\varepsilon^2\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1-\varepsilon^4\frac{1}{D^2}(\boldsymbol{c}_0\cdot\boldsymbol{D})(\boldsymbol{c}_0\cdot\boldsymbol{L})\eta_1+\chi(\boldsymbol{D})\Big(d_\alpha\eta_1^2+B(\boldsymbol{D})\eta_1^2+\underline{O}(\varepsilon^2|||\eta_1|||^2)\Big)=0,$$

and applying Lemma 4.1(iv) one can further simplify it to

$$g(\boldsymbol{D})\eta_1 + 2\varepsilon^2\eta_1 + \chi(\boldsymbol{D})\Big(d_\alpha\eta_1^2 + B(\boldsymbol{D})\eta_1^2 + \underline{O}(\varepsilon^3|||\eta_1|||) + \underline{O}(\varepsilon^2|||\eta_1|||^2)\Big) = 0.$$

The reduction is completed by introducing the KP scaling

$$\eta_1(x,y) = \varepsilon^2\zeta(\varepsilon x, \varepsilon^2 y),$$

noting that $I : \eta_1 \to \zeta$ is an isomorphism $\mathcal{X}_1 \to Y_1^\varepsilon$ and $\chi(\boldsymbol{D})L^2(\mathbb{R}^2) \to Y_0^\varepsilon$ and choosing $R > 1$ large enough so that $\zeta_k^\star \in B_R(0)$ (and $\varepsilon > 0$ small enough so that $B_R(0) \subseteq Y_1^\varepsilon$ is contained in $I[X_1]$). Here we have replaced $(\chi(\boldsymbol{D})H^1(\mathbb{R}^2), \|\cdot\|_1)$ and $(\chi_\varepsilon(\boldsymbol{D})L^2(\mathbb{R}^2), \|\cdot\|_0)$ by the identical spaces $(\chi(\boldsymbol{D})L^2(\mathbb{R}^2), \|\cdot\|_0)$ and $(Y_0^\varepsilon, \|\cdot\|_{Y_0})$ in order to work exclusively with the scales $\{Y_s, \|\cdot\|_{Y^s}\}_{s\geq 0}$ and $\{Y_s^\varepsilon, \|\cdot\|_{Y_s}\}_{s\geq 0}$ of function spaces. We find that $\zeta \in B_R(0) \subseteq Y_1^\varepsilon$ satisfies the equation

$$\varepsilon^{-2}g_\varepsilon(\boldsymbol{D})\zeta + 2\zeta + d_\alpha\chi_\varepsilon(\boldsymbol{D})\zeta^2 + \chi_\varepsilon(\boldsymbol{D})B_\varepsilon(\boldsymbol{D})\zeta^2 + \underline{\mathcal{O}}_0^\varepsilon(\varepsilon^{\frac{1}{2}}\|\zeta\|_{Y_1}) = 0, \tag{75}$$

which now holds in Y_0^ε, where

$$g_\varepsilon(\boldsymbol{k}) = g(\varepsilon k_1, \varepsilon^2 k_2), \qquad B_\varepsilon(k_1,k_2) = B(\varepsilon k_1, \varepsilon^2 k_2)$$

and the symbol $\underline{\mathcal{O}}_n^\varepsilon(\varepsilon^s\|\zeta\|_{Y_1}^r)$ denotes a smooth function $\mathcal{R} : B_R(0) \subseteq Y_1^\varepsilon \to Y_n^\varepsilon$ which satisfies the estimates

$$\|\mathcal{R}(\zeta)\|_{Y_n} \lesssim \varepsilon^s\|\zeta\|_{Y_1}^r, \quad \|\mathrm{d}\mathcal{R}[\zeta]\|_{L(Y_1,Y_n)} \lesssim \varepsilon^s\|u\|_{Y_1}^{r-1}$$

for each $\zeta \in B_R(0) \subseteq Y_1^\varepsilon$ (with $r \geq 1$, s, $n \geq 0$). Note that $|||\eta_1|||^2 = \varepsilon\|\zeta\|_{Y_1}^2$ and that the change of variables from (x,y) to $(\varepsilon x, \varepsilon^2 y)$ introduces a further factor of $\varepsilon^{\frac{3}{2}}$ in the remainder term. The invariance of the reduced equation under $\eta_1(x,y) \mapsto \eta_1(-x,-y)$ is inherited by (75), which is invariant under the reflection $\zeta(x,y) \mapsto \zeta(-x,-y)$.

5 Solution of the reduced equation

In this section we find solitary-wave solutions of the reduced equation (75), noting that in the formal limit $\varepsilon \to 0$ it reduces to the stationary KP-I equation

$$-(\beta - \beta_0)\zeta_{xx} + 2\zeta + \sec^2\tfrac{1}{2}\alpha\frac{D_2^2}{D_1^2}\zeta + d_\alpha\zeta^2 = 0,$$

which has explicit solitary-wave solutions $\zeta_k^\star$. For this purpose we use a perturbation argument, rewriting (75) as a fixed-point equation and applying the following version of the implicit-function theorem.

Theorem 5.1 *Let $\mathcal{W}$ be a Banach space, W_0 and Λ_0 be open neighbourhoods of respectively $w^\star$ in $\mathcal{W}$ and the origin in $\mathbb{R}$, and $\mathcal{H}\colon W_0 \times \Lambda_0 \to \mathcal{W}$ be a function which is differentiable with respect to $w \in W_0$ for each $\lambda \in \Lambda_0$. Furthermore, suppose that $\mathcal{H}(w^\star, 0) = 0$, $\mathrm{d}_1\mathcal{H}[w^\star, 0]\colon \mathcal{W} \to \mathcal{W}$ is an isomorphism,*

$$\lim_{w\to w^\star}\|\mathrm{d}_1\mathcal{H}[w,0] - \mathrm{d}_1\mathcal{H}[w^\star,0]\|_{L(\mathcal{W},\mathcal{W})} = 0$$

and

$$\lim_{\lambda\to 0}\|\mathcal{H}(w,\lambda) - \mathcal{H}(w,0)\|_{\mathcal{W}} = 0, \quad \lim_{\lambda\to 0}\|\mathrm{d}_1\mathcal{H}[w,\lambda] - \mathrm{d}_1\mathcal{H}[w,0]\|_{L(\mathcal{W},\mathcal{W})} = 0$$

uniformly over $w \in W_0$.

There exist open neighbourhoods $W \subseteq W_0$ of $w^\star$ in $\mathcal{W}$ and $\Lambda \subseteq \Lambda_0$ of the origin in $\mathbb{R}$, and a uniquely determined mapping $h\colon \Lambda \to W$ with the properties that

(i) h is continuous at the origin with $h(0) = w^\star$,

(ii) $\mathcal{H}(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,

(iii) $w = h(\lambda)$ whenever $(w, \lambda) \in W \times \Lambda$ satisfies $\mathcal{H}(w, \lambda) = 0$.

Theorem 5.2 *For each sufficiently small value of* $\varepsilon > 0$ *equation* (75) *has a solution* ζ_k^ε *in* $Y_{1+\theta}^\varepsilon$ *with* $\zeta(x,y) = \zeta(-x-y)$ *for all* $(x,y) \in \mathbb{R}^2$ *and* $\|\zeta_k^\varepsilon - \zeta_k^\star\|_{Y_{1+\theta}} \to 0$ *as* $\varepsilon \to 0$.

The first step in the proof of Theorem 5.2 is to write (75) as the fixed-point equation

$$\zeta + \varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} \left(d_\alpha \chi_\varepsilon(\boldsymbol{D})\zeta^2 + \chi_\varepsilon(\boldsymbol{D}) B_\varepsilon(\boldsymbol{D})\zeta^2 + \underline{\mathcal{O}}_0^\varepsilon(\varepsilon^{\frac{1}{2}}\|\zeta\|_{Y_1})\right) = 0 \tag{76}$$

and use the following result to 'replace' the nonlocal operator with the KP operator

$$L_\alpha = 2 - (\beta - \beta_0)\partial_x^2 + \sec^2 \tfrac{1}{2}\alpha \, \frac{D_2^2}{D_1^2}.$$

Proposition 5.3 *Suppose that* $\theta \in [0,1]$. *The inequality*

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})} - \frac{1}{2 + (\beta-\beta_0)k_1^2 + \sec^2 \frac{1}{2}\alpha \, \frac{k_2^2}{k_1^2}} \right| \lesssim \frac{\varepsilon^{1-\theta}}{(1 + |(k_1, \frac{k_2}{k_1})|^2)^{\frac{1}{2}(1+\theta)}}$$

holds uniformly over $|k_1|, \frac{|k_2|}{|k_1|} < \delta/\varepsilon$.

Proof. Clearly

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})} - \frac{1}{2 + (\beta-\beta_0)k_1^2 + \sec^2 \frac{1}{2}\alpha \, \frac{k_2^2}{k_1^2}} \right|$$
$$= \frac{\left|\tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1}) - (\beta-\beta_0)\varepsilon^2 k_1^2 - \sec^2 \frac{1}{2}\alpha \, \varepsilon^2 \frac{k_2^2}{k_1^2}\right|}{\left(2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})\right)\left(2 + (\beta-\beta_0)k_1^2 + \sec^2 \frac{1}{2}\alpha \, \frac{k_2^2}{k_1^2}\right)}$$

furthermore

$$\left|\tilde{g}\left(s_1, \frac{s_2}{s_1}\right) - (\beta-\beta_0)s_1^2 - \sec^2 \tfrac{1}{2}\alpha \, \frac{s_2^2}{s_1^2}\right| \lesssim \left|\left(s_1, \frac{s_2}{s_1}\right)\right|^3,$$

and

$$\tilde{g}\left(s_1, \frac{s_2}{s_1}\right) \gtrsim \left|\left(s_1, \frac{s_2}{s_1}\right)\right|^2$$

for $|s_1|, \frac{|s_2|}{|s_1|} \leq \delta$ and sufficiently small δ (see Remark 5.1).

It follows that

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})} - \frac{1}{2 + (\beta-\beta_0)k_1^2 + \sec^2 \frac{1}{2}\alpha \, \frac{k_2^2}{k_1^2}} \right| \lesssim \frac{\varepsilon|(k_1, \frac{k_2}{k_1})|^3}{(1 + |(k_1, \frac{k_2}{k_1})|^2)^2}$$
$$\lesssim \frac{\varepsilon}{(1 + |(k_1, \frac{k_2}{k_1})|^2)^{\frac{1}{2}}}$$

uniformly over $|k_1|, \frac{|k_2|}{|k_1|} < \delta/\varepsilon$, and the stated result follows from this inequality and the observation that $\varepsilon \lesssim \delta(1+t^2)^{-\frac{1}{2}}$ when $|t| < \delta/\varepsilon$. □

Lemma 5.4 *Suppose that* $\theta \in [0,1]$. *The estimates*

$$\varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} \underline{\mathcal{O}}_0^\varepsilon(\varepsilon^{\frac{1}{2}}\|\zeta\|_{Y_1}) = \underline{\mathcal{O}}_{1+\theta}^\varepsilon(\varepsilon^{\frac{1}{2}}\|\zeta\|_{Y_{1+\theta}}),$$
$$\varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} B_\varepsilon(\boldsymbol{D})\zeta^2 = \underline{\mathcal{O}}_{1+\theta}^\varepsilon(\varepsilon^{\frac{1}{2}-\frac{\theta}{2}}\|\zeta\|_{1+\theta}^2)$$

and

$$\left(\varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} - L_\alpha^{-1}\right) \chi_\varepsilon(\boldsymbol{D})\zeta^2 = \underline{\mathcal{O}}_{1+\theta}^\varepsilon(\varepsilon^{1-\theta}\|\zeta\|_{1+\theta}^2)$$

hold for all $\zeta \in Y_{1+\theta}^\varepsilon$.

Proof. It follows from Proposition 5.3 (with $\theta = 1$) that

$$\frac{\varepsilon^2}{2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})} \lesssim \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{-1},$$

from which the first estimate is an immediate consequence (note that $\|\zeta\|_1 \leq \|\zeta\|_{1+\theta}$). Furthermore

$$\begin{aligned}
&\left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}+\frac{\theta}{2}} \frac{\varepsilon^2}{2\varepsilon^2 + \tilde{g}(\varepsilon k_1, \varepsilon \frac{k_2}{k_1})} B(\varepsilon k_1, \varepsilon^2 k_2) \\
&\lesssim \left(1 + k_1^2 + \frac{k_2^2}{k_1^2}\right)^{-\frac{1}{2}+\frac{\theta}{2}} \frac{\varepsilon \left|\frac{k_2}{k_1}\right|}{1 + \varepsilon^2 \frac{k_2^2}{k_1^2}} \\
&= \varepsilon^{\frac{1}{2}-\frac{\theta}{2}} \left(\frac{\left|\frac{k_2}{k_1}\right|}{1 + k_1^2 + \frac{k_2^2}{k_1^2}} \frac{1}{1 + \varepsilon^2 \frac{k_2^2}{k_1^2}} \right)^{\frac{1}{2}-\frac{\theta}{2}} \left(\frac{\varepsilon \left|\frac{k_2}{k_1}\right|}{1 + \varepsilon^2 \frac{k_2^2}{k_1^2}} \right)^{\frac{1}{2}+\frac{\theta}{2}} \\
&\lesssim \varepsilon^{\frac{1}{2}-\frac{\theta}{2}},
\end{aligned}$$

such that

$$\left\| \varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} B_\varepsilon(\boldsymbol{D}) \zeta\xi \right\|_{Y_{1+\theta}} \lesssim \varepsilon^{\frac{1}{2}-\frac{\theta}{2}} \|\zeta\xi\|_0 \leq \varepsilon^{\frac{1}{2}-\frac{\theta}{2}} \|\zeta\|_{L^4(\mathbb{R}^2)} \|\xi\|_{L^4(\mathbb{R}^2)} \lesssim \varepsilon^{\frac{1}{2}-\frac{\theta}{2}} \|\zeta\|_{Y_{1+\theta}} \|\xi\|_{Y_{1+\theta}}.$$

for all $\zeta, \xi \in Y_{1+\theta}^\varepsilon$ (see Proposition 1.7(i)).

The final estimate follows from the observation that

$$\left\| \left(\varepsilon^2 \left(g_\varepsilon(\boldsymbol{D}) + 2\varepsilon^2\right)^{-1} - L_\alpha^{-1} \right) \chi_\varepsilon(\boldsymbol{D}) \zeta\xi \right\|_{Y_{1+\theta}} \lesssim \varepsilon^{1-\theta} \|\zeta\xi\|_0 \lesssim \varepsilon^{1-\theta} \|\zeta\|_{Y_{1+\theta}} \|\xi\|_{Y_{1+\theta}}$$

for all $\zeta, \xi \in Y_{1+\theta}^\varepsilon$, in which the first inequality follows from Proposition 5.3. □

Using the above lemma, one can write equation (76) as

$$\zeta + F_\varepsilon(\zeta) = 0,$$

in which

$$F_\varepsilon(\zeta) = d_\alpha L_\alpha^{-1} \chi_\varepsilon(\boldsymbol{D}) \zeta^2 + \underline{\mathcal{O}}_{1+\theta}^\varepsilon(\varepsilon^{\frac{1}{2}-\frac{\theta}{2}} \|\zeta\|_{1+\theta}).$$

It is convenient to replace this equation with

$$\zeta + \tilde{F}_\varepsilon(\zeta) = 0,$$

where $\tilde{F}_\varepsilon(\zeta) = F_\varepsilon(\chi_\varepsilon(\boldsymbol{D})\zeta)$ and study it in the fixed space $Y_{1+\theta}$ for $\theta \in (\frac{1}{2}, 1)$ (the solution sets of the two equations evidently coincide); we choose $\theta > \frac{1}{2}$ so that $Y_{1+\theta}$ is embedded in $C_\mathrm{b}(\mathbb{R}^2)$ and $\theta < 1$ so that the remainder term in $\tilde{F}_\varepsilon(\zeta)$ vanishes at $\varepsilon = 0$.

We establish Theorem 5.2 by applying Theorem 5.1 with

$$\mathcal{W} = Y_{1+\theta}^\mathrm{e} := \{\zeta \in Y_{1+\theta} : \zeta(x, y) = \zeta(-x, -y) \text{ for all } (x, y) \in \mathbb{R}^2\},$$

$W_0 = B_R(0) \subseteq Y_{1+\theta}$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of ε_0, and

$$\mathcal{H}(\zeta, \varepsilon) := \zeta + \tilde{F}_{|\varepsilon|}(\zeta)$$

(here ε is replaced by $|\varepsilon|$ so that $\mathcal{H}(\zeta, \varepsilon)$ is defined for ε in a full neighbourhood of the origin in $\mathbb{R}$).

We begin by verifying that the functions $\zeta_k^\star$ belong to $Y_{1+\theta}^\mathrm{e}$.

Proposition 5.5 *Each lump solution $\zeta_k^\star$ belongs to Y_2.*

Proof. First note that $(\zeta_k^\star)^2$ belongs to $L^2(\mathbb{R}^2) = Y_0$ because $|\zeta_k^\star(x, y)| \lesssim (1 + x^2 + y^2)^{-1}$ for all $(x, y) \in \mathbb{R}^2$ (see Lemma 1.1(i)). Since $\zeta_k^\star$ satisfies

$$\zeta_k^\star + L_\alpha^{-1}(\zeta_k^\star)^2 = 0$$

and L_α^{-1} is a regularising operator of order 2 for the scale $\{Y_r, \|\cdot\|_{Y_r}\}_{r\geq 0}$, one finds that $\zeta_k^\star \in Y_2$. □

Observe that $\mathcal{H}(\cdot, \varepsilon)$ is a continuously differentiable function $B_R(0) \subseteq Y_{\mathrm{e}}^{1+\theta} \to Y_{\mathrm{e}}^{1+\theta}$ for each fixed $\varepsilon \geq 0$, so that

$$\lim_{\zeta \to \zeta_k^\star} \|\mathrm{d}_1\mathcal{H}[\zeta, 0] - \mathrm{d}_1\mathcal{H}[\zeta_k^\star, 0]\|_{L(Y_{1+\theta}, Y_{1+\theta})} = 0.$$

The facts that

$$\lim_{\varepsilon \to 0} \|\mathcal{H}(\zeta, \varepsilon) - \mathcal{H}(\zeta, 0)\|_{Y_{1+\theta}} = 0, \quad \lim_{\varepsilon \to 0} \|\mathrm{d}_1\mathcal{H}[\zeta, \varepsilon] - \mathrm{d}_1\mathcal{H}[\zeta, 0]\|_{L(Y_{1+\theta}, Y_{1+\theta})} = 0$$

uniformly over $\zeta \in B_R(0) \subseteq Y_{\mathrm{e}}^{1+\theta}$ are obtained from the equation

$$\mathcal{H}(\zeta, \varepsilon) - \mathcal{H}(\zeta, 0) = L_\alpha^{-1}\left(\chi_\varepsilon(\mathbf{D})\left(\chi_\varepsilon(\mathbf{D})\zeta\right)^2 - \zeta^2\right) + \underline{\mathcal{O}}_{1+\theta}^{\varepsilon}(\varepsilon^{\frac{1}{2}-\frac{\theta}{2}}\|\zeta\|_{1+\theta})$$

using Corollary 5.8 below, which is a consequence of the next two lemmas.

Lemma 5.6 *Fix $\theta > \frac{1}{2}$. The estimate*

$$\|L_\alpha^{-1}\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\xi)\big)\|_{Y_{1+\theta}} \lesssim \varepsilon\|\zeta\|_{Y_{1+\theta}}\|\xi\|_{Y_{1+\theta}}$$

holds for all $\zeta, \xi \in Y_{1+\theta}$.

Proof. Recall that L_α^{-1} is a regularising operator of order 2 for the scale $\{Y_r, \|\cdot\|_{Y_r}\}_{r \geq 0}$ and that $\chi_\varepsilon(\mathbf{D})$ is a bounded projection on all subspaces of $L^2(\mathbb{R}^2)$. It follows that

$$\begin{aligned}
&\|L_\alpha^{-1}\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\xi)\big)\|_{Y_{1+\theta}} \\
&\qquad \leq \|\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\xi)\big)\|_0 \\
&\qquad \leq \|((\chi_\varepsilon(\mathbf{D}) + I)\zeta)((\chi_\varepsilon(\mathbf{D}) - I)\xi)\|_0 \\
&\qquad \leq \|(\chi_\varepsilon(\mathbf{D}) + I)\zeta\|_\infty\|(\chi_\varepsilon(\mathbf{D}) - I)\xi\|_0 \\
&\qquad \lesssim \|(\chi_\varepsilon(\mathbf{D}) + I)\zeta\|_{Y_{1+\theta}}\|(\chi_\varepsilon(\mathbf{D}) - I)\xi\|_0 \\
&\qquad \leq 2\|\zeta\|_{Y_{1+\theta}}\|(\chi_\varepsilon(\mathbf{D}) - I)\xi\|_0,
\end{aligned}$$

where we have used the embedding $Y_{1+\theta} \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^2)$. To estimate $\|(\chi_\varepsilon(\mathbf{D}) - I)\zeta\|_0$, note that

$$\mathbb{R}^2 \setminus C_\varepsilon \subset \underbrace{\left\{(k_1, k_2) \colon |k_1| > \frac{\delta}{\varepsilon}\right\}}_{= C_\varepsilon^1} \cup \underbrace{\left\{(k_1, k_2) \colon \left|\frac{k_2}{k_1}\right| > \frac{\delta}{\varepsilon}\right\}}_{= C_\varepsilon^2},$$

so that

$$\begin{aligned}
\|(\chi_\varepsilon(\mathbf{D}) - I)\zeta\|_0^2 &= \int_{\mathbb{R}^2 \setminus C_\varepsilon} \|\hat{\zeta}\|^2 \,\mathrm{d}k \\
&\leq \int_{C_\varepsilon^1} \|\hat{\zeta}\|^2 \,\mathrm{d}k + \int_{C_\varepsilon^2} \|\hat{\zeta}\|^2 \,\mathrm{d}k \\
&\leq \frac{\varepsilon^2}{\delta^2}\int_{C_\varepsilon^1} k_1^2\|\hat{\zeta}\|^2 \,\mathrm{d}k + \frac{\varepsilon^2}{\delta^2}\int_{C_\varepsilon^2} \frac{k_2^2}{k_1^2}\|\hat{\zeta}\|^2 \,\mathrm{d}k \\
&\leq \frac{2\varepsilon^2}{\delta^2}\|\zeta\|_{Y_1}^2.
\end{aligned}$$

□

Lemma 5.7 *Fix $\theta \in (0, 1)$. The estimate*

$$\|L_\alpha^{-1}(\chi_\varepsilon(\mathbf{D}) - I)(\zeta\xi)\|_{Y_{1+\theta}} \lesssim \varepsilon^{1-\theta}\|\zeta\|_{Y_1}\|\xi\|_{Y_1} \leq \varepsilon^{\frac{1}{2}-\frac{\theta}{2}}\|\zeta\|_{Y_{1+\theta}}\|\xi\|_{Y_{1+\theta}},$$

holds for all $\zeta, \xi \in Y_{1+\theta}$.

Proof. For $\nu \in \{k_1, \frac{k_2}{k_1}\}$ we find that

$$\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{1+\theta}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-2}|\nu|^{2-2\theta} = \left(\frac{\nu^2}{1+k_1^2+\frac{k_2^2}{k_1^2}}\right)^{1-\theta} \leq 1,$$

so that

$$\begin{aligned}
&\|L_\alpha^{-1}(\chi_\varepsilon(\mathbf{D})-I)\zeta\xi\|_{Y_{1+\theta}}^2 \\
&\quad\lesssim \int_{C_\varepsilon^1\cup C_\varepsilon^2}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{1+\theta}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-2}|\mathcal{F}[\zeta\xi]|^2\,\mathrm{d}k \\
&\quad\lesssim \left(\frac{\varepsilon}{\delta}\right)^{2-2\theta}\int_{C_\varepsilon^1}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{1+\theta}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-2}|k_1|^{2-2\theta}|\mathcal{F}[\zeta\xi]|^2\,\mathrm{d}k \\
&\qquad+\left(\frac{\varepsilon}{\delta}\right)^{2-2\theta}\int_{C_\varepsilon^2}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{1+\theta}\left(1+k_1^2+\frac{k_2^2}{k_1^2}\right)^{-2}\left|\frac{k_2}{k_1}\right|^{2-2\theta}|\mathcal{F}[\zeta\xi]|^2\,\mathrm{d}k \\
&\quad\leq \left(\frac{\varepsilon}{\delta}\right)^{2-2\theta}\|\zeta\xi\|_0^2 \\
&\quad\lesssim \left(\frac{\varepsilon}{\delta}\right)^{2-2\theta}\|\zeta\|_{L^4(\mathbb{R}^2)}^2\|\xi\|_{L^4(\mathbb{R}^2)}^2 \\
&\quad\lesssim \left(\frac{\varepsilon}{\delta}\right)^{2-2\theta}\|\zeta\|_{Y_1}^2\|\xi\|_{Y_1}^2,
\end{aligned}$$

where we have used Parseval's theorem, the Cauchy-Schwarz inequality and the embedding $Y_1 \hookrightarrow L^4(\mathbb{R}^2)$. □

Corollary 5.8 *Fix* $\theta \in (\frac{1}{2}, 1)$. *The estimate*

$$\left\|L_\alpha^{-1}\Big(\chi_\varepsilon(\mathbf{D})\big((\chi_\varepsilon(\mathbf{D})\zeta)(\chi_\varepsilon(\mathbf{D})\xi)\big)-\zeta\xi\Big)\right\|_{Y_{1+\theta}} \lesssim \varepsilon^{1-\theta}\|\zeta\|_{Y_{1+\theta}}\|\xi\|_{Y_{1+\theta}}$$

holds for all ζ, $\xi \in Y_{1+\theta}$.

Proof. This result is obtained by writing

$$\begin{aligned}
&L_\alpha^{-1}\Big(\chi_\varepsilon(\mathbf{D})\big((\chi_\varepsilon(\mathbf{D})\zeta)(\chi_\varepsilon(\mathbf{D})\xi)\big)-\zeta\xi\Big) \\
&\quad= \tfrac{1}{2}L_\alpha^{-1}\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D})+1)\zeta)((\chi_\varepsilon(\mathbf{D})-1)\xi)\big) \\
&\qquad+\tfrac{1}{2}L_\alpha^{-1}\chi_\varepsilon(\mathbf{D})\big(((\chi_\varepsilon(\mathbf{D})+1)\xi)((\chi_\varepsilon(\mathbf{D})-1)\zeta)\big)+L_\alpha^{-1}(\chi_\varepsilon(\mathbf{D})-1)(\zeta\xi),
\end{aligned}$$

and applying Lemma 5.6 to the first two terms on the right-hand side and Lemma 5.7 to the third. □

It thus remains to show that

$$\mathrm{d}_1\mathcal{H}[\zeta_k^\star, 0] = I + 2d_\alpha L_\alpha^{-1}(\zeta_k^\star\cdot)$$

is an isomorphism; this fact follows from the following result.

Lemma 5.9 *The operator* $L_\alpha^{-1}(\zeta_k^\star\cdot) : Y_{1+\theta} \to Y_{1+\theta}$ *is compact.*

Proof. Let $\{\zeta_j\}$ be a sequence which is bounded in Y_1. We can find a subsequence of $\{\zeta_j\}$ (still denoted by $\{\zeta_j\}$) which converges weakly in $L^2(\mathbb{R}^2)$ (because $\{\zeta_j\}$ is bounded in $L^2(\mathbb{R}^2)$) and strongly in $L^2(|\mathbf{x}| < n)$ for each $n \in \mathbb{N}$ (by Proposition 1.7(ii) and a 'diagonal' argument). Denote the limit by ζ_∞. Since

$$\|\zeta_k^\star\zeta_j - \zeta_k^\star\zeta_\infty\|_{L^2(|\mathbf{x}|<n)} \leq \|\zeta_k^\star\|_\infty\|\zeta_j-\zeta_\infty\|_{L^2(|\mathbf{x}|<n)} \to 0$$

as $j \to \infty$ for each $n \in \mathbb{N}$ and

$$\sup_j \|\zeta_k^\star\zeta_j\|_{L^2(|\mathbf{x}|>n)} \leq \sup_{|\mathbf{x}|>n}|\zeta_k^\star(\mathbf{x})|\sup_j\|\zeta_j\|_0 \to 0$$

as $n \to \infty$ we conclude that $\{\zeta_k^\star \zeta_j\}$ converges to $\zeta_k^\star \zeta_\infty$ in $L^2(\mathbb{R}^2)$ as $j \to \infty$. It follows that $\zeta \mapsto \zeta_k^\star \zeta$ is compact $Y_1 \to L^2(\mathbb{R})$ and hence $Y_{1+\theta} \to L^2(\mathbb{R})$; the result follows from this fact and the observation that L_α^{-1} is continuous $L^2(\mathbb{R}^2) \to Y_{1+\theta}$. □

Lemma 5.10 *The operator* $I + 2d_\alpha L_\alpha^{-1}(\zeta_k^\star \cdot)$ *is an isomorphism* $Y_{1+\theta} \to Y_{1+\theta}$.

Proof. The previous result shows that $I + 2d_\alpha L_\alpha^{-1}(\zeta_k^\star \cdot) : Y_{1+\theta} \to Y_{1+\theta}$ is Fredholm with index 0; it therefore remains to show that it is injective.

Suppose that $\zeta \in Y_{1+\theta}$ satisfies

$$\zeta + 2d_\alpha L_\alpha^{-1}(\zeta_k^\star \zeta) = 0. \tag{77}$$

It follows that

$$k_1 \hat{\zeta} = \frac{-2d_\alpha k_1^3}{2k_1^2 + (\beta - \beta_0)k_1^4 + \sec^2 \frac{1}{2}\alpha \, k_2^2} \mathcal{F}[\zeta_k^\star \zeta], \qquad k_2 \hat{\zeta} = \frac{-2d_\alpha k_1 k_2}{2k_1^2 + (\beta - \beta_0)k_1^4 + \sec^2 \frac{1}{2}\alpha \, k_2^2} \mathcal{F}[\zeta_k^\star \zeta]$$

and hence $\zeta \in H^{n+1}(\mathbb{R}^2)$ whenever $\zeta_k^\star \zeta \in H^n(\mathbb{R}^2)$. Since $\zeta \in L^2(\mathbb{R}^2)$ and $\zeta \in H^m(\mathbb{R}^2)$ implies $\zeta_k^\star \zeta \in H^m(\mathbb{R}^2)$ we find by bootstrapping that $\zeta \in H^\infty(\mathbb{R}^2)$.

Since ζ is smooth and satisfies (77) it satisfies the linear equation

$$\big((\beta - \beta_0)\zeta_{xx} + 2\zeta + 2d_\alpha(\zeta_k^\star \zeta)\big)_{xx} - \sec^2 \tfrac{1}{2}\alpha \, \zeta_{zz} = 0,$$

and the only smooth solution to this equation with $\zeta(x, y) = \zeta(-x, -y)$ for all $(x, y) \in \mathbb{R}^2$ is the trivial solution (see Lemma 1.1(iii)). □

To establish Theorem 1.2 it remains to confirm that the formula

$$\eta = \eta_1 + \eta_2(\eta_1), \quad \eta_1(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y)$$

leads to the estimate

$$\eta(x, y) = \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + o(\varepsilon^2)$$

uniformly over $(x, y) \in \mathbb{R}^2$. This fact follows from the calculations

$$\|\zeta_k^\varepsilon - \zeta_k^\star\|_\infty \lesssim \|\zeta_k^\varepsilon - \zeta_k^\star\|_{Y_{1+\theta}} = o(1),$$

such that

$$\begin{aligned}\eta_1(x, y) &= \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + \varepsilon^2(\zeta_k^\varepsilon - \zeta_k^\star)(\varepsilon x, \varepsilon^2 y) \\ &= \varepsilon^2 \zeta_k^\star(\varepsilon x, \varepsilon^2 y) + o(1)\end{aligned}$$

uniformly in (x, y), and

$$\|\eta_2(\eta_1)\|_\infty \lesssim \|\eta_2(\eta_1)\|_3 \lesssim \varepsilon |||\eta_1|||^2 \lesssim \varepsilon^3$$

by Theorem 3.7 and $|||\eta_1||| = \varepsilon \|\zeta\|_{Y_1}$ with $\zeta \in B_R(0) \subseteq Y_{1+\theta}^\varepsilon$.

Appendix A Dispersion relation

Recall the dispersion relation

$$g(\boldsymbol{k}) = 0, \tag{78}$$

where

$$g(\boldsymbol{k}) = -\frac{1}{|\boldsymbol{k}|^2}(\alpha(\boldsymbol{c}_0 \cdot \boldsymbol{k}^\perp)(\boldsymbol{c}_0 \cdot \boldsymbol{k}) + \mathsf{c}(|\boldsymbol{k}|^2)(\boldsymbol{c}_0 \cdot \boldsymbol{k})^2) + 1 + \beta|\boldsymbol{k}|^2$$

is an analytic function $\tilde{g}$ of k_1 and $\frac{k_2}{k_1}$ with $\tilde{g}(0, 0) = 0$ if

$$\boldsymbol{c}_0 = \begin{pmatrix} c_0 \cos \frac{1}{2}\alpha \\ -c_0 \sin \frac{1}{2}\alpha \end{pmatrix}, \qquad c_0^2 = \tfrac{2}{\alpha} \tan \tfrac{1}{2}\alpha.$$

Suppose that $\boldsymbol{k} \neq \boldsymbol{0}$, so that (78) is equivalent to

$$\mathsf{c}(|\boldsymbol{k}|^2) = \kappa(|\boldsymbol{k}|^2, \theta),$$

where

$$\kappa(\mu, \theta) = \frac{1 + \beta\mu - \alpha c_0^2 \sin\theta \cos\theta}{c_0^2 \cos^2\theta}$$

and θ is the angle between $\boldsymbol{c}_0$ and $\boldsymbol{k}$ (note that $g(\boldsymbol{k}) > 0$ if $\cos\theta = 0$). The function $\kappa(\mu, \cdot)$ takes every value in $[\kappa_{\min}(\mu), \infty)$, where

$$\kappa_{\min}(\mu) = \frac{1 + \beta\mu}{c_0^2} - \frac{c_0^2 \alpha^2}{4(1 + \beta\mu)},$$

and the minimum is attained at

$$\theta = -\tan^{-1} \frac{\alpha c_0^2}{2(1 + \beta\mu)}.$$

It follows that $g(\boldsymbol{k}) \neq 0$ for all $\boldsymbol{k}$ with given magnitude $|\boldsymbol{k}|$ if and only if $\mathsf{c}(|\boldsymbol{k}|^2) < \kappa_{\min}(|\boldsymbol{k}|^2)$.

The functions c and $\kappa_{\min}$ are both strictly increasing and concave on $[0, \infty)$ with $\mathsf{c}(0) = \kappa_{\min}(0)$. Obviously $\mathsf{c}(\mu) < \kappa_{\min}(\mu)$ for $\mu \in (0, \infty)$ if $\mathsf{c}'(\mu) < \kappa_{\min}'(\mu)$ for $\mu \in [0, \infty)$, and since

$$\mathsf{c}'(\mu) \leq \mathsf{c}'(0) = \frac{1}{2}\left(-\frac{\cot\alpha}{\alpha} + \operatorname{cosec}^2\alpha\right)$$

(because c is concave) and

$$\kappa_{\min}'(\mu) = \frac{1}{2}\alpha\beta\left(\frac{1}{(1 + \beta\mu)^2} + \cot^2\tfrac{1}{2}\alpha\right)\tan\tfrac{1}{2}\alpha > \tfrac{1}{2}\alpha\beta\cot\tfrac{1}{2}\alpha$$

this condition is met if

$$\frac{1}{2}\left(-\frac{\cot\alpha}{\alpha} + \operatorname{cosec}^2\alpha\right) < \tfrac{1}{2}\alpha\beta\cot\tfrac{1}{2}\alpha,$$

that is if

$$\beta > \beta^\star := \tfrac{1}{\alpha}\left(-\tfrac{1}{\alpha}\cot\alpha + \operatorname{cosec}^2\alpha\right)\tan\tfrac{1}{2}\alpha.$$

Remark 5.1 *The calculation*

$$\tilde{g}(k_1, \tfrac{k_2}{k_1}) = \left(\beta + \frac{1}{2\alpha^2}(\cos\alpha - \alpha\operatorname{cosec}\alpha)\right)k_1^2 + \sec^2\tfrac{1}{2}\alpha\,\tfrac{k_2^2}{k_1^2} + \underline{O}(|(k_1, \tfrac{k_2}{k_1})|^3)$$

as $(k_1, \frac{k_2}{k_1}) \to (0,0)$ *shows that* $(0,0)$ *is a strict local minimum of* $\tilde{g}$ *if*

$$\beta > \beta_0 := \frac{1}{2\alpha^2}(-\cos\alpha + \alpha\operatorname{cosec}\alpha).$$

Note that

$$\beta^\star - \beta_0 = \frac{1}{\alpha^2}\operatorname{cosec}^3\alpha\sin^4\tfrac{1}{2}\alpha(2\alpha - \sin 2\alpha) \geq 0$$

with equality if and only if $\alpha = 0$ *(the common value is* $\frac{1}{3}$*).*

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