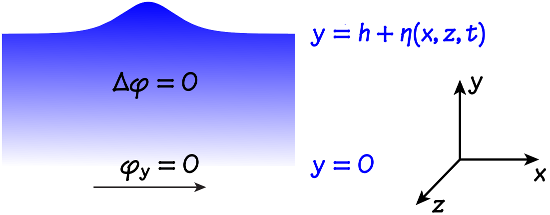
THE WATER-WAVE PROBLEM



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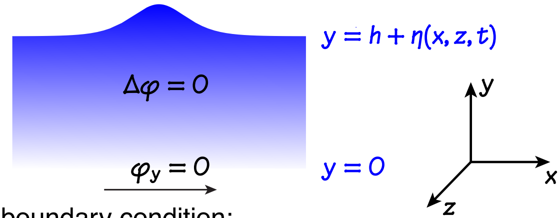

Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

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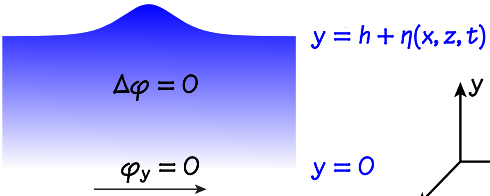

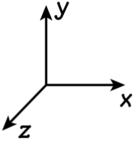

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Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \sigma\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

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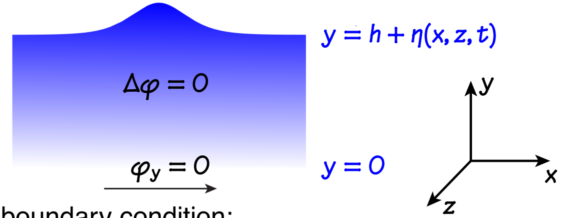

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Difficulties:

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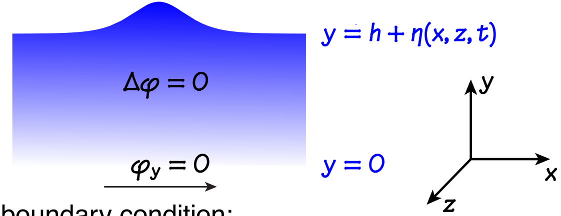

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Difficulties:

- A free-boundary value problem

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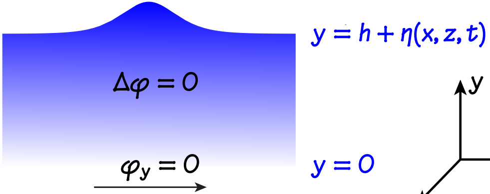

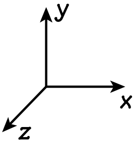

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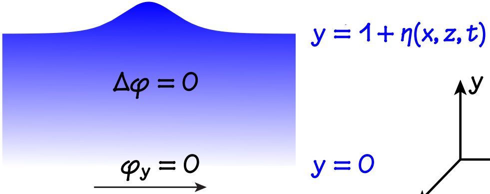

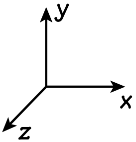

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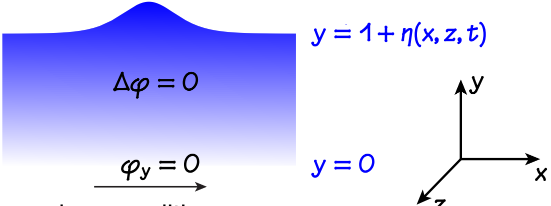

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$$-c\eta_x = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

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Difficulties:

- A free-boundary value problem
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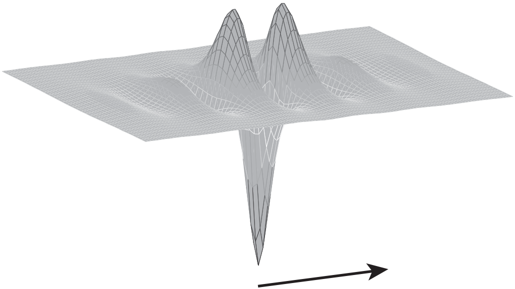
Parameter: $\beta = \sigma/gh^2$

Solitary waves: $\eta(x,z,t) = \eta(x-ct,z)$, $\eta(x-ct,z) \to 0$ as $|(x-ct,z)| \to \infty$

FULLY LOCALISED SOLITARY WAVES

- Weak surface tension ($\beta < 1/3$):



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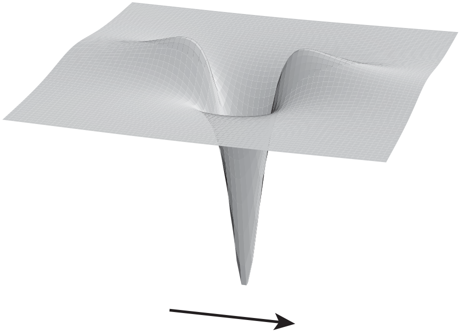

MODELLING

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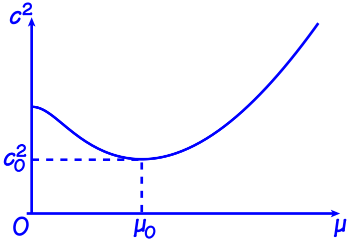
- Dispersion relation for linear wave trains $\eta \sim \cos\mu(x - ct)$:



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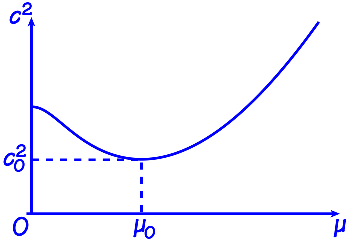

- The Ansatz

$$c^2 = c_0^2(1 - \varepsilon^2), \qquad \eta(x,z) = \varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{i\mu_0 x} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-i\mu_0 x}\right) + O(\varepsilon^2)$$

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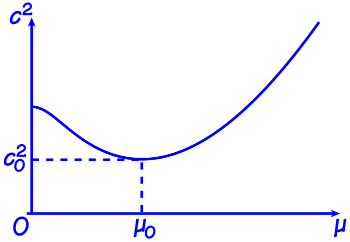

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leads to the Davey-Stewartson equation

$$\zeta - \zeta_{xx} - \zeta_{zz} - |\zeta|^2\zeta - \zeta\Delta^{-1}\partial_x^2|\zeta|^2 = 0$$

THE DS EQUATION

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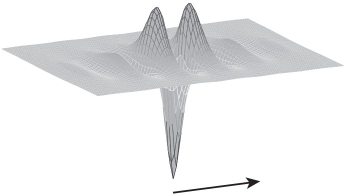

$\mathrm{Re}\,\zeta(x,z)e^{i\mu_0 x}$

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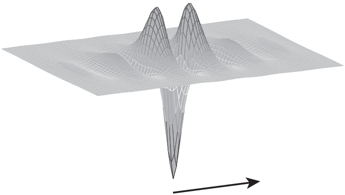

$\mathrm{Re}\,\zeta(x,z)e^{i\mu_0 x}$

- This solution is a critical point of the functional

$$\mathcal{T}_0(\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(|\zeta_x|^2 + |\zeta_z|^2 + |\zeta|^2) - \frac{1}{4}|\zeta|^4 - \frac{1}{4}|\zeta|^2\Delta^{-1}\partial_x^2|\zeta|^2\right\}dx\,dz$$

with function space

$$X = \overline{C_0^\infty(\mathbb{R}^2)} = H^1(\mathbb{R}^2)$$

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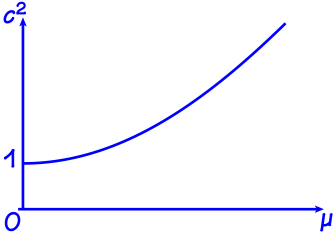

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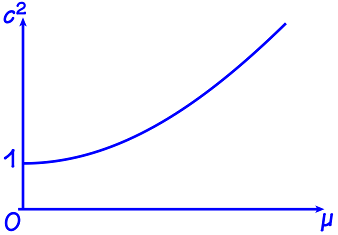

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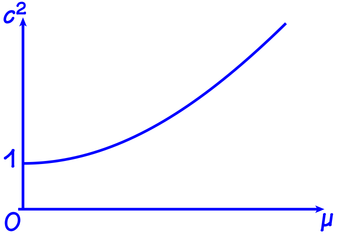

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leads to the Kadomtsev-Petviashvili equation

$$\zeta_{xx} - \zeta - \frac{3}{2}\zeta^2 - \partial_x^{-2}\zeta_{zz} = 0$$

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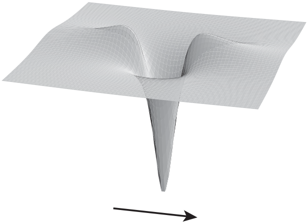
- An explicit solitary-wave solution

$$\zeta(x,z) = -8\frac{3 - x^2 + z^2}{(3 + x^2 + z^2)^2}$$



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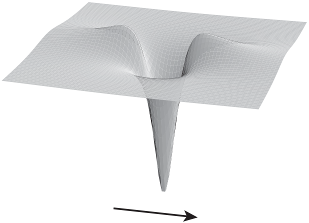

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$$\mathcal{I}_0(\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(\zeta^2 + (\partial_x^{-1}\zeta_z)^2 + \zeta_x^2) - \frac{1}{3}\zeta^3\right\} dx\,dz$$

with function space

$$X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

VARIATIONAL PRINCIPLE

- Luke's variational principle

$$\delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{0}^{1+\eta} \left(-c\varphi_x + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \right) dy + \frac{1}{2}\eta^2 + \beta\left(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1\right) \right\} dx\, dz = 0$$

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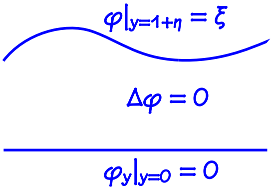
recovers the hydrodynamic equations

- Use a Dirichlet-Neumann operator:

$$\delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -c\eta_x \xi + \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \beta\left(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1\right) \right\} dx\, dz = 0$$

where $\xi = \varphi|_{y=1+\eta}$ and

$$G(\eta)\xi = \sqrt{1 + \eta_x^2 + \eta_z^2}\, \varphi_n|_{y=1+\eta}$$



FORMULATION

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$$J(\eta) = \mathcal{F}(\eta, \xi(\eta)) = \mathcal{K}(\eta) - c^2\mathcal{L}(\eta),$$

where

$$\mathcal{K}(\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta\right\}dx\,dz,$$

$$\mathcal{L}(\eta) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta K(\eta)\eta\,dx\,dz, \qquad K(\eta)\xi := -\partial_x(G(\eta)^{-1}\xi_x)$$

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- Study the new variational problem $J'(\eta) = 0$

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- Study the new variational problem $\mathcal{J}'(\eta) = 0$
- $K : H^3(\mathbb{R}^2) \to \mathcal{B}(H^{5/2}(\mathbb{R}^2), H^{3/2}(\mathbb{R}^2))$ is analytic at the origin

REDUCTION ($\beta < 1/3$)

- Find critical points of $\mathcal{J}(\eta) = \mathcal{K}(\eta) - c^2\mathcal{L}(\eta)$

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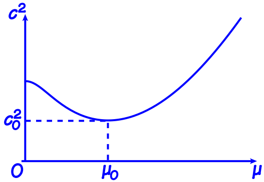

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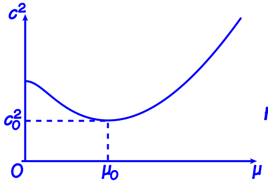

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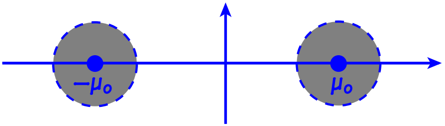
- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1-\chi(D))\eta,$$

where χ is the characteristic function of this set:



$$|\mu - \mu_0| < \delta$$

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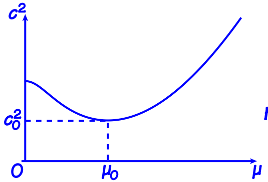

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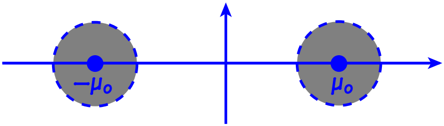

$$|\mu - \mu_0| < \delta$$

- $J'(\eta) = 0 \quad \Rightarrow$

$$\chi(D)J'(\eta_1+\eta_2) = 0,$$
$$(1-\chi(D))J'(\eta_1+\eta_2) = 0$$

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- Find critical points of $\mathcal{J}(\eta) = \mathcal{K}(\eta) - c^2\mathcal{L}(\eta)$
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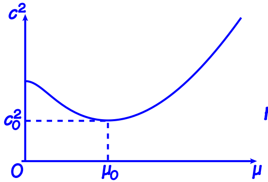


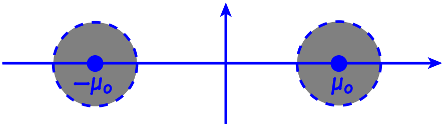
$$c^2 = c_0^2(1-\varepsilon^2),$$

$$\eta(x,z) = \varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{i\mu_0 x} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-i\mu_0 x}\right) + O(\varepsilon^2)$$

- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1-\chi(D))\eta,$$

where χ is the characteristic function of this set:



$$|\mu - \mu_0| < \delta$$

- $\mathcal{J}'(\eta) = 0 \quad \Rightarrow$

$$\chi(D)\mathcal{J}'(\eta_1+\eta_2) = 0,$$
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Solve for $\eta_2 = \eta_2(\eta_1)$, set $\mathfrak{I}(\eta_1) = \mathcal{J}(\eta_1 + \eta_2(\eta_1))$, consider $\mathfrak{I}'(\eta_1) = 0$

REDUCTION ($\beta > 1/3$)

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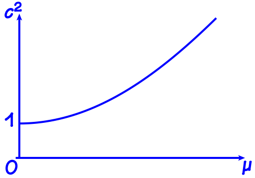

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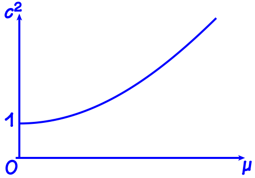

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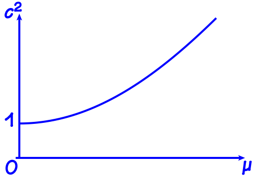

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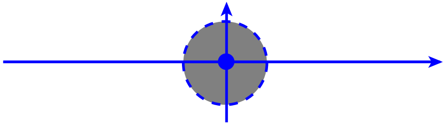

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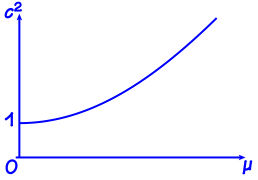

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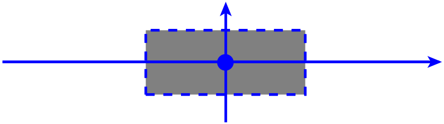

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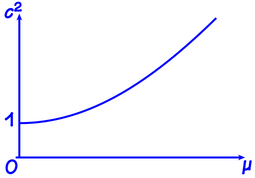

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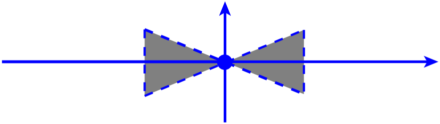

$$|\mu_1| < \delta, \quad \left|\frac{\mu_2}{\mu_1}\right| < \delta$$

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- Study this functional in

$$B_R(0) \subseteq X_\varepsilon := \chi(\varepsilon D_1, \varepsilon^2 D_2) X, \qquad X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

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- Look for minimisers of $\tilde{I}_0$ over N

GEOMETRICAL INTERPRETATION

$$N = \{\zeta \neq 0 : \langle \tilde{\Gamma}'_0(\zeta), \zeta \rangle = 0\}$$

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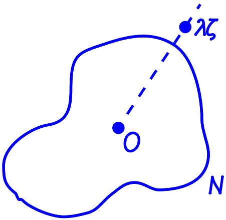
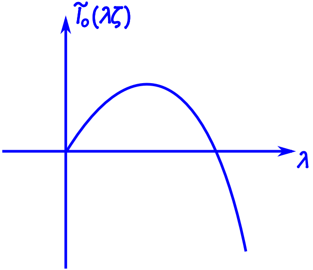
$$N = \{\zeta \neq 0 : \langle \tilde{I}_0'(\zeta), \zeta \rangle = 0\}$$

Any ray

$$\{\lambda\zeta : K(\zeta) > 0, \lambda > 0\}$$

intersects N in precisely one point and the value of $\tilde{I}_0$ along such a ray attains a strict maximum at this point





EXISTENCE THEORY

How to find a minimiser for $\tilde{\mathcal{I}}_0(\zeta) = \frac{1}{2}\|\zeta\|^2 - \mathcal{K}(\eta)$ over

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  – Key: Weak convergence of $\{\zeta_n\}$ implies convergence of $\{\mathcal{K}(\zeta_n)\}$