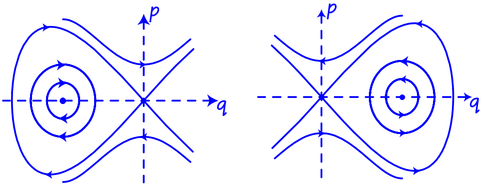
THE (HAMILTONIAN) KIRCHGÄSSNER REDUCTION

Mark Groves

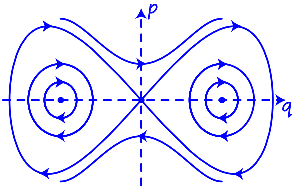
Homoclinic orbits in Hamiltonian systems



$$q_x = \partial_p H = q,$$
$$p_x = -\partial_q H = \mu q - Cq^2$$

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}\mu q^2 + Cq^3$$

$$q(x) = \frac{3\mu}{2C}\,\mathrm{sech}^2\,\frac{\sqrt{\mu}x}{2}$$



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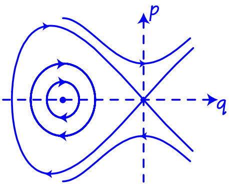
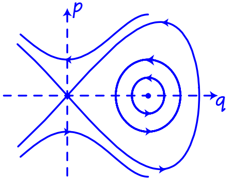
Homoclinic orbits in Hamiltonian systems





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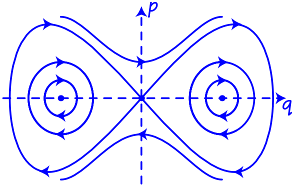

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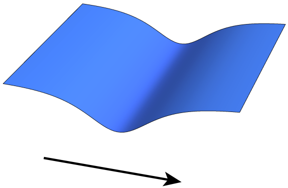
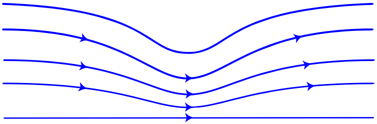
- Persistence by reversibility

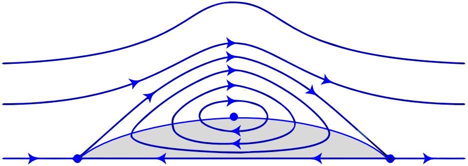
Solitary water waves

- Wave of depression with strong surface tension (Kirchgässner 1988)



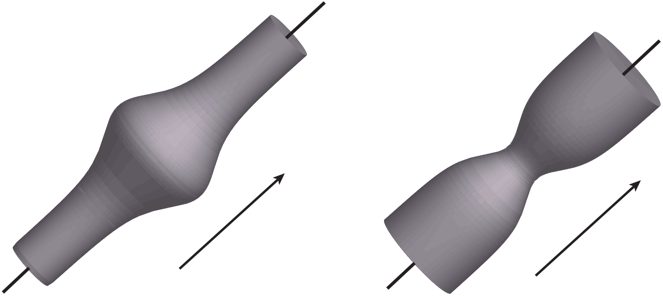


- Gravity wave of elevation 'riding' a flow with large constant vorticity (Kozlov, Kuznetsov & Lokharu 2020)



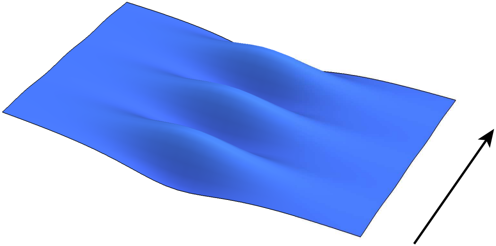
Solitary waves on a ferrofluid jet

- Axisymmetric surface waves on a ferrofluid jet surrounding a current-carrying wire (Groves & Nilsson 2018)



Three-dimensional travelling water waves

- Periodic gravity-capillary travelling water waves with a localised transverse profile (Groves 2001)



Framework

Begin with a spatial dynamics problem

$$\begin{aligned} u_x &= f^\mu(u) && \text{in } \Omega, \\ B^\mu(u) &= 0 && \text{on } \partial\Omega \end{aligned}$$

- The domain Ω is fixed
- Reversible: there is a reverser S with

$$f^\mu(Su) = -f^\mu(u), \qquad B^\mu(Su) = \pm B^\mu(u)$$

- Hamiltonian: there is a Hamiltonian system (X, Ω, H^μ) with

$$\Omega^\mu|_u(f^\mu(u), v) + \Upsilon(B^\mu(u), v) = dH^\mu[u](v)$$

  for all v
- The formal linearisation $L = df^0[0]$ is a Riesz spectral operator with finite-dimensional centre subspace

Step 1: Linearise the boundary conditions

$$u_x = f^\mu(u) \qquad \text{in } Q,$$
$$B^\mu(u) = 0 \qquad \text{on } \partial Q$$

Use a near-identity change of variables

$$u = v + G^\mu(v), \qquad G^\mu(v) = O(|\mu|\|v\| + \|v\|^2)$$

to linearise and de-parametrise the boundary conditions:

$$v_x = L(v) + N^\mu(v) \qquad \text{in } Q,$$
$$L_B(v) = 0 \qquad \text{on } \partial Q,$$

where $L = df^0[0]$ and $L_B = dB^0[0]$

- Hamiltonian: (X, Q_{lbc}, H_{lbc}) with $H^\mu_{lbc}(v) = H^\mu(v + G^\mu(v))$ and $Q^0_{lbc}|_0 = Q^0|_0$

Step 2: Centre-manifold reduction

$$v_x = L(v) + N^\mu(v) \qquad \text{in } \mathcal{Q},$$
$$L_B(v) = 0 \qquad \text{on } \partial\mathcal{Q}$$

- Write $X = X_c \oplus X_h$
- All small globally bounded solutions lie on

$$M_c = \{(v_c, v_h) : v_h = h^\mu(v_c)\},$$

  where $h^\mu(v_c) = O(|\mu|\|v_c\| + \|v_c\|^2)$

- Reduced equation:

$$v_{cx} = Lv_c + N^\mu(v_c + h^\mu(v_c))$$

  - Hamiltonian: $(X_c, \Omega_{cm}, H_{cm})$ with $H^\mu_{cm}(v_c) = H^\mu_{lbc}(v_c + h^\mu(v_c))$ and $\Omega^0_{cm}|_0 = \Omega^0_{lbc}|_0$

Step 3: Coordinates for the centre manifold

- Use a symplectic basis for X_c:
  - Write $v_c = qe + pf$, where $Le = 0$, $Lf = e$ with $\Omega^0|_0(e,f) = 1$:

$$\Omega^0_{cm}|_0(v_c^1, v_c^2) = \Omega_{can}(v_c^1, v_c^2) = q^1p^2 - p^1q^2$$

- Use a near-identity 'Darboux' change of variables

$$v_c = w_c + D^\mu(w_c), \qquad D^\mu(w_c) = O(|\mu|\|w_c\| + \|w_c\|^2)$$

  to transform $(X_c, \Omega_{cm}, H_{cm})$ into $(X_c, \Omega_{can}, H_{red})$, **where** $H^\mu_{red}(w_c) = H^\mu_{cm}(w_c + D^\mu(w_c))$

- Possibly use a further near-identity '(partial) normal-form' transformation that does not change Ω_{can}
- Final result:

$$q_x = \partial_p H^\mu_{red}(q,p),$$
$$p_x = -\partial_q H^\mu_{red}(q,p)$$

  - Reversible: $S(q,p) = (q,-p)$ if $Se = e$, $Sf = -f$

How to calculate H_{red}?

- $$\begin{aligned} u &= v + G^\mu(v) \\ &= v_c + h^\mu(v_c) + G^\mu(v_c + h^\mu(v_c)) \\ &= w_c + D^\mu(w_c) + h^\mu(w_c + D^\mu(w_c)) \\ &\qquad + G^\mu(w_c + D^\mu(w_c) + h^\mu(w_c + D^\mu(w_c))) \\ &=: w_c + k^\mu(w_c), \qquad k^\mu(w_c) = O(|\mu|\|w_c\| + \|w_c\|^2) \end{aligned}$$
- $H^\mu_{red}(w_c) = H^\mu(w_c + k^\mu(w_c))$
- Coefficients in the Maclaurin series

$$k^\mu(w_c) = \sum_{\substack{i+j_1+j_2\geq 2 \\ j_1+j_2\geq 1}} k_{i,j_1,j_2}\mu^i q^{j_1} p^{j_2}$$

can be calculated by substituting $u = w_c + k^\mu(w_c)$ into

$$u_x = f^\mu(u) \qquad \text{in } \Omega,$$
$$B^\mu(u) = 0 \qquad \text{on } \partial\Omega$$

- Simplify using

$$\Omega^\mu|_u(f^\mu(u), v) + \Upsilon(B^\mu(u), v) = dH^\mu[u](v)$$

How to calculate H_{red}?

- The coefficient of q^3 in H^{μ}_{red} is

$$C_3 = H^0_3(e, e, e)$$

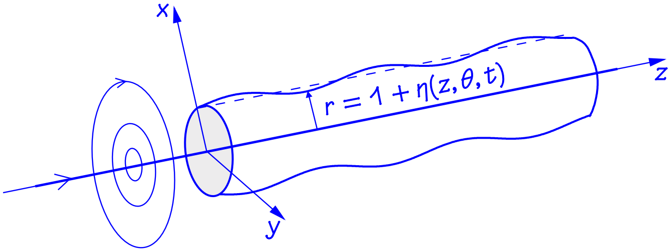
- If $C_3 = 0$ the coefficient of q^4 in H^{μ}_{red} is

$$C_4 = H^0_4(e, e, e, e) + \tfrac{3}{2} H^0_3(e, e, k_{0,20}),$$

where

$$f^0_1 k_{0,20} = -f^0_2(e, e),$$
$$B^0_1 k_{0,20} = -B^0_2(e, e)$$

Solitary waves on a ferrofluid jet



- Axisymmetric travelling waves:
  - $\eta(z, \theta, t) = \eta(z - ct)$
  - The hydrodynamic problem decouples
- Formulate in terms of (η, ϕ)

Solitary waves on a ferrofluid jet

The governing equations

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad 0 < r < 1 + \eta(z),$$

$$\phi_r = 0, \qquad r = 0,$$

$$\eta_z + \phi_r - \phi_z \eta_z = 0, \qquad r = 1 + \eta(z),$$

$$-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - a\frac{T'(\eta)}{1+\eta} + \beta\left(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\right) = 0, \qquad r = 1 + \eta(z)$$

follow from the formal variational principle

$$\delta \int \left\{ \int_0^{1+\eta} \left(\frac{1}{2} r\phi_r^2 + \frac{1}{2} r\phi_z^2 - r\phi_z \right) dr - aT(\eta) + \beta(1+\eta)(1+\eta_z^2)^{1/2} - \frac{1}{2}\beta(1+\eta)^2 \right\} dz = 0$$

Solitary waves on a ferrofluid jet

- 'Flatten' the equations with the transformation

$$s = \frac{r}{1+\eta(z)}, \qquad \Phi(s,z) = \phi(r,z)$$

- The flattened equations follow from the variational principle

$$\delta \int L(\eta,\Phi;\eta_z,\Phi_z)\, dz = 0,$$

where

$$L(\eta,\Phi;\eta_z,\Phi_z) =$$
$$\int_0^1 \left\{ \frac{1}{2}\left(s\Phi_s^2 + \left[\Phi_z - \frac{s\eta_z\Phi_s}{1+\eta}\right]^2 (1+\eta)^2 s \right) - \left(\Phi_z - \frac{s\eta_z\Phi_s}{1+\eta}\right)(1+\eta)^2 s \right\} ds$$
$$- aT(\eta) + \beta(1+\eta)(1+\eta_z^2)^{1/2} - \frac{1}{2}\beta(1+\eta)^2$$

Solitary waves on a ferrofluid jet

- Variational principle: $\delta \int L(\eta, \Phi; \eta_z, \Phi_z)\, dz = 0$
- Legendre transform:

$$\omega = \frac{\delta L}{\delta \eta_z}, \xi = \frac{\delta L}{\delta \Phi_z} \quad \Rightarrow \quad \eta_z = \eta_z(\eta, \omega, \Phi, \xi), \Phi_z = \Phi_z(\eta, \omega, \Phi, \xi)$$

and

$$H(\eta, \omega, \Phi, \xi) = \eta_z \omega + \int_0^1 s \Phi_z \xi\, ds - L(\eta, \Phi, \eta_z, \Phi_z)$$
$$= \int_0^1 \left\{ \frac{1}{2} \left[\frac{\xi}{(1+\eta)^2} + 1 \right]^2 (1+\eta)^2 s - \frac{1}{2} s \Phi_s^2 \right\} ds$$
$$+ aT(\eta) - (1+\eta)\sqrt{\beta^2 - W^2} + \frac{1}{2}\beta(1+\eta)^2,$$

where $W = \frac{1}{1+\eta}\left(\omega + \frac{1}{1+\eta}\int_0^1 s^2 \Phi_s \xi\, ds\right)$ satisfies $|W| < \beta$

- $\Omega\big((\eta_1, \Phi_1, \omega_1, \xi_1), (\eta_2, \Phi_2, \omega_2, \xi_2)\big) = \omega_2 \eta_1 - \omega_1 \eta_2 + \int_0^1 (\xi_2 \Phi_1 - \xi_1 \Phi_2) r\, dr$
- Reversibility: $S(\eta, \omega, \Phi, \xi) = (\eta, -\omega, -\Phi, \xi)$

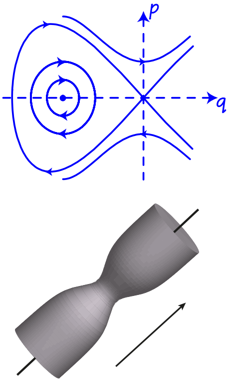
Solitary waves on a ferrofluid jet

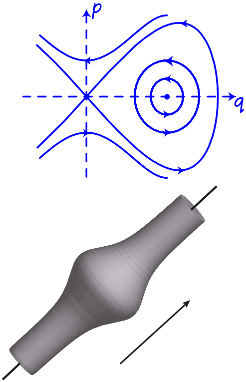
- Write

$$\beta > \tfrac{1}{4}, \quad a = 2 + \beta + \mu$$

- After scaling z and q one obtains the reduced Hamiltonian

$$H^{\mu}_{red} = \tfrac{1}{2}p^2 - \tfrac{1}{2}\mu q^2 + \tfrac{1}{6}(aT'''(0) - 6)q^3 + O(\mu|(q,p)|^2 + |(q,p)|^4)$$





Solitary waves on a ferrofluid jet

- $$H(\eta,\omega,\Phi,\xi) = \eta_z\omega + \int_0^1 s\Phi_z\xi\,ds - L(\eta,\Phi,\eta_z,\Phi_z)$$
  $$= \int_0^1 \left\{\frac{1}{2}\left[\frac{\xi}{(1+\eta)^2}+1\right]^2(1+\eta)^2 s - \frac{1}{2}s\Phi_s^2\right\} ds$$
  $$+ aT(\eta) - (1+\eta)\sqrt{\beta^2 - W^2} + \frac{1}{2}\beta(1+\eta)^2,$$
  where $W = \frac{1}{1+\eta}\left(\omega + \frac{1}{1+\eta}\int_0^1 s^2\Phi_s\xi\,ds\right)$ satisfies $|W| < \beta$
- $$Q\big((\eta_1,\Phi_1,\omega_1,\xi_1),(\eta_2,\Phi_2,\omega_2,\xi_2)\big) = \omega_2\eta_1 - \omega_1\eta_2 + \int_0^1 (\xi_2\Phi_1 - \xi_1\Phi_2)s\,ds$$
- We deal with the case $\beta = 0$ by restricting to the submanifold

$$\omega = -\frac{1}{1+\eta}\int_0^1 s^2\Phi_s\xi\,ds$$

Solitary waves on a ferrofluid jet

- $$H(\eta,\Phi,\xi) = \int_0^1 \left\{ \frac{1}{2}\left[\frac{\xi}{(1+\eta)^2}+1\right]^2 (1+\eta)^2 s - \frac{1}{2} s\Phi_s^2 \right\} ds + aT(\eta)$$

- $$\Omega|_{(\eta,\Phi,\xi)}\big((\eta_1,\Phi_1,\xi_1),(\eta_2,\Phi_2,\xi_2)\big) = \int_0^1 (\xi_2\Phi_1 - \Phi_2\xi_1) s\, ds$$
$$- \frac{\eta_1}{\eta}\int_0^1 s^2(\Phi_{2z}\xi + \Phi_z\xi_2)\, ds + \frac{\eta_2}{\eta}\int_0^1 s^2(\Phi_{1z}\xi + \Phi_z\xi_1)\, ds$$

- Reversibility: $S(\eta,\Phi,\xi) = (\eta,-\Phi,\xi)$