MODEL EQUATIONS

- Whitham equation:

$$u_t + (n(u) + Lu)_x = 0$$

$$\widehat{(Lu)}(k) = \underbrace{\left(\frac{\tanh(k)}{k}\right)^{\frac{1}{2}}}_{= m(k)} \hat{u}(k), \qquad n(u) = u^2 + \cdots$$

- KdV equation:

$$Lu = 1 + \tfrac{1}{6}u'', \qquad n(u) = u^2$$

Travelling waves: $u(x,t) = u(x - \nu t)$

Formal weakly nonlinear theory for Whitham gives KdV:

- Long-wave expansion:

$$\nu = 1 + \mu^{\gamma}\nu_{lw}, \qquad u(x) = \mu^{a} w(\mu^{\beta} x),$$

where $\mu = \frac{1}{2}\int u^2 := \mathcal{Q}(u)$

- Choosing $a = \frac{2}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, we find that

$$\mu^{\frac{4}{3}}\left(\tfrac{1}{6}w'' - \nu_{lw} w + w^2\right) + o(\mu^{\frac{4}{3}}) = 0$$

KdV has stable solitary-wave solutions. Does Whitham?

VARIATIONAL PRINCIPLES

KdV:

- Solitary waves are local minimisers of

$$\mathcal{E}_{\mathrm{lw}}(w) := \int \left(\tfrac{1}{12}(w')^2 - \tfrac{1}{3}w^3\right), \qquad \mathcal{Q}(w) = 1$$

- Semilinear structure
- A nonempty set of minimisers over H^1, minimising sequences converge (Albert, Zeng)

Whitham:

- Solitary waves are local minimisers of

$$\mathcal{E}(u) := \int \left(-\tfrac{1}{2}uLu - N(u)\right), \qquad \mathcal{Q}(u) = \mu$$

- Linear part is smoothing: $L : H^s \longrightarrow H^{s+\frac{1}{2}}$
- Look for minimisers over

$$U = \{u \in H^1 : \|u\|_1 < R\}$$

- Minimising sequences with $\sup \|u_n\|_1 < R$ converge

PERIODIC WAVES

- Minimise

$$\mathcal{E}_P(u) := \int_{-\frac{P}{2}}^{\frac{P}{2}} \left(-\tfrac{1}{2}uLu - N(u)\right), \qquad Q_P(u) = \mu$$

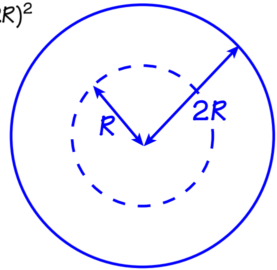
over

$$U_P = \{u \in H^1_P : \|u\|_1 < R\}$$

- We regularise and penalise the functional:

$$\mathcal{E}_{P,\rho}(u) = \mathcal{E}_P(u) + \rho(\|u\|_1^2)$$

  - ρ is smooth and increasing
  - $\rho(t) = 0$ for $0 \leq t < R^2$
  - $\rho(t) \to \infty$ as $t \uparrow (2R)^2$



- Look for minimisers of $\mathcal{E}_{P,\rho}$ over $\{\|u\|_1 < 2R\}$

MINIMISATION PROCEDURE

$$\mathcal{E}_{P,\rho}(u) = \underbrace{\mathcal{E}_P(u)}_{\substack{\text{Defined}\\ \text{on } H_P^s}} + \underbrace{\rho(\|u\|_1^2),}_{\substack{\text{Defined}\\ \text{on } H_P^1}} \qquad s \in (\tfrac{1}{2}, 1)$$

- $\mathcal{E}_{P,\rho}$ has a minimiser $u_P \neq 0$, since it is
  - weakly lower-semicontinuous
  - coercive ($\|u\|_1 \to 2R \quad \Rightarrow \quad \mathcal{E}_{P,\rho}(u) \to \infty$)

  on $\{u \in H_P^1 : \mathcal{Q}_P(u) = \mu\}$

- u_P lies in the region unaffected by the penalisation:
  - A priori estimates show that

$$\mathcal{E}_{P,\rho}(u_P) < -\mu, \qquad \mathcal{E}'_{P,\rho}(u_P) + \nu_P \mathcal{Q}'(u_P) = 0$$

$$\Rightarrow \quad \nu_P > 1 - \varepsilon, \quad \|u_P\|_1^2 \leq c\mu$$

  - Take $w \in C_0^\infty$ and $\mathcal{Q}(w) = 1$:

$$\mathcal{E}_P(\mu^{\frac{2}{3}} w(\mu^{\frac{1}{3}} x)) = -\mu + \mu^{\frac{5}{3}} \mathcal{E}_{\mathrm{lw}}(w) + o(\mu^{\frac{5}{3}})$$

  - Take $w(x) = \sqrt{\lambda}\psi(\lambda x)$, $\lambda \ll 1$:

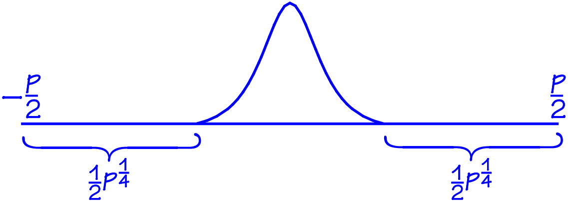
$$\mathcal{E}_{\mathrm{lw}}(w) = \int \left(\tfrac{1}{12}\lambda^2(\psi')^2 - \tfrac{1}{3}\lambda^{\frac{1}{2}}\psi^3\right) < 0$$

SPECIAL MINIMISING SEQUENCE

- Let $\{v_P\}_P$ be a bounded family of functions in H^1 with

$$\mathit{supp}(v_P) \subset (-\tfrac{P}{2}, \tfrac{P}{2}), \quad \mathit{dist}(\pm\tfrac{P}{2}, \mathit{supp}(v_P)) \geq \tfrac{1}{2}P^{\frac{1}{4}}$$

and let $w_P \in H^1_P$ be its periodic extension



- $$\mathcal{E}(v_P) - \mathcal{E}_P(w_P) \to 0, \qquad \mathcal{Q}(v_P) = \mathcal{Q}_P(w_P)$$

$$\|\mathcal{E}'(v_P) - \mathcal{E}'_P(w_P)\|_{H^1(-\frac{P}{2},\frac{P}{2})}, \|\mathcal{E}'(v_P)\|_{H^1(\{|x|>\frac{P}{2}\})} \to 0$$

$$\|\mathcal{Q}'(v_P) - \mathcal{Q}'_P(w_P)\|_{H^1(-\frac{P}{2},\frac{P}{2})}, \|\mathcal{Q}'(v_P)\|_{H^1(\{|x|>\frac{P}{2}\})} = 0$$

- We translate and truncate u_{P_n} to obtain a minimising squence $\{\tilde{u}_n\}$ for $\mathcal{E}$ with

$$\|\tilde{u}_n\|_1^2 \leq c\mu, \qquad \|\mathcal{E}'(\tilde{u}_n) + \nu_n \mathcal{Q}'(\tilde{u}_n)\|_1 \to 0$$

CONCENTRATION-COMPACTNESS

A minimising sequence $\{u_n\}$ with $\mathcal{E}(u_n) \to I_\mu$, $\mathcal{Q}(u_n) = \mu$, $\sup \|u_n\|_1 < R$ can

- concentrate ($\{u_n\}$ behaves like a minimising sequence for the periodic problem)
  - leads to convergence to a minimiser u
  - $\nu > 1$ (supercritical), $\|u\|_1^2 = O(\mu)$
  - $\|u\|_2 \lesssim \|u\|_1$ (regularity theory)
- vanish (the wave dissolves into ripples)
  - easily ruled out
- dichotomise (the wave splits into two parts)
  - Dichotomy is ruled out by strict sub-additivity

$$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2}$$

    (easy to prove when $\mathcal{N}(u)$ is homogeneous)

- Try to approximate $\mathcal{N}(u_n)$ with $\frac{1}{3}\int u_n^3$
  - $\mathcal{E}(u_n) < -\mu - c\mu^{\frac{5}{3}} \quad \Rightarrow \quad \mathcal{N}(u_n) \leq -c\mu^{\frac{5}{3}}$
  - So we require $\mathcal{N}_r(u_n) = o(\mu^{\frac{5}{3}})$ (not clear for a general minimising sequence)
  - $\mathcal{N}_r(u_n) = O(\|u_n\|_\infty^2 \|u_n\|_0^2) = O(\|u_n\|_1^4) = O(\mu^2)$

SCALING

- Do solutions scale like KdV ($u(x) = \mu^{\frac{2}{3}} w(\mu^{\frac{1}{3}} x)$)?
- Show that $|||u|||_\tau^2 = O(\mu)$:
  - $|||v|||_\tau^2 := \int \left(v^2 + \mu^{-\frac{4\tau}{3}} (v'')^2 \right), \qquad \tau < 1$
  - $\|v\|_\infty \lesssim \mu^{\frac{\tau}{6}} |||v|||_\tau$
- Split a solution of $\mathcal{F}^{-1}[(\nu - m(k))\hat{u}] = n(u)$:

$$\hat{u}_1(k) := \xi(k)\hat{u}(k), \qquad \hat{u}_2(k) := (1 - \xi(k))\hat{u}(k)$$

$$\begin{aligned} (\nu - m)\hat{u}_1 &= \xi \mathcal{F}[n(u)], \\ (\nu - m)\hat{u}_2 &= (1 - \xi)\mathcal{F}[n(u)] \end{aligned}$$

- Estimates:
  - $\|u_2''\|_0 \lesssim \|(n(u))''\|_0 \leq c\|u\|_\infty \|u''\|_0$
  - $(\nu - m) \gtrsim k^2$ for $|k| < k_0$, so that

$$\begin{aligned} \int |u_1''|^2 &\lesssim \int (\nu - m)^2 |\hat{u}_1|^2 \\ &\lesssim \|n(u)\|_0^2 \\ &\lesssim \|u\|_\infty^2 \|u\|_0^2 \end{aligned}$$

  - $\Rightarrow \|u''\|_0^2 \lesssim \|u\|_2^2 \|u\|_\infty$

SCALING

$$|||v|||_\tau^2 := \int \left(v^2 + \mu^{-\frac{4\tau}{3}} (v'')^2 \right)$$

- We know that

$$\int |u''|^2 \lesssim \|u\|_2^2 \|u\|_\infty^2 \lesssim \mu^{1+\frac{\tau}{3}} |||u|||_\tau^2$$

- Multiply by $\mu^{-\frac{4\tau}{3}}$, add $\int u^2 = 2\mu$:

$$\mu^{-1} |||u|||_\tau^2 \lesssim \left(1 + \mu^{1-\tau} (\mu^{-1} |||u|||_\tau^2)\right)$$

- $Q := \{\tau \in [0,1) : \|u\|_\tau^2 \lesssim \mu\}$
  - $0 \in Q$
  - $\tau \in Q \Rightarrow [0,\tau] \subset Q$
  - Suppose $\tau_\star := \sup Q < 1$. Choose $\varepsilon > 0$ so that $\tau_\star + \frac{11}{3}\varepsilon < 1$:

$$\mu^{-1} |||u|||_{\tau_\star+\varepsilon}^2 \lesssim \left(1 + \mu^{1-\tau_\star-\frac{11}{3}\varepsilon} \underbrace{(\mu^{-1} |||u|||_{\tau_\star-\varepsilon}^2)}_{\lesssim 1}\right)$$

CONVERGENCE

- Scaling arguments yield

$$\mathcal{E}(\mu^{\frac{2}{3}}w(\mu^{\frac{1}{3}}x)) = -\mu + \mu^{\frac{5}{3}}\mathcal{E}_{\mathrm{lw}}(w) + o(\mu^{\frac{5}{3}}), \qquad w \in D_{\mathrm{lw}}$$

$$\mathcal{E}(u) = -\mu + \mu^{\frac{5}{3}}\mathcal{E}_{\mathrm{lw}}(\mu^{-\frac{2}{3}}u(\mu^{-\frac{1}{3}}x))) + o(\mu^{\frac{5}{3}}), \qquad u \in D_{\mu}$$

- Thus
  - $I_\mu = -\mu + \mu^{\frac{5}{3}} I_{\mathrm{lw}} + o(\mu^{\frac{5}{3}})$
  - $I_\mu = -\mu + \mu^{\frac{5}{3}} \mathcal{E}_{\mathrm{lw}}(\mu^{-\frac{2}{3}}u(\mu^{-\frac{1}{3}}x))) + o(\mu^{\frac{5}{3}})$
  - $\mathcal{E}_{\mathrm{lw}}(\mu^{-\frac{2}{3}}u(\mu^{-\frac{1}{3}}x))) = I_{\mathrm{lw}} + o(1)$
- It follows that

$$\sup_{u\in D_\mu} \mathrm{dist}_{H^1}\left(\mu^{-\frac{2}{3}}u(\mu^{-\frac{1}{3}}x), D_{\mathrm{lw}}\right) \longrightarrow 0$$

Otherwise:

- There exist $\{\mu_n\}$ and $\{u_n\} \in D_{\mu_n}$ with

$$\mu_n \longrightarrow 0,$$

$$\inf_{w\in D_{\mathrm{lw}}} \|\mu^{-\frac{2}{3}}u_n(\mu^{-\frac{1}{3}}x) - w\|_1 \geq \varepsilon$$

- $\{\mu_n^{-\frac{2}{3}}u_n(\mu_n^{-\frac{1}{3}}x)\}$ is a minimising sequence for $\mathcal{E}_{\mathrm{lw}}$ over $\{w \in H^1 : Q(w) = 1\}$, so converges