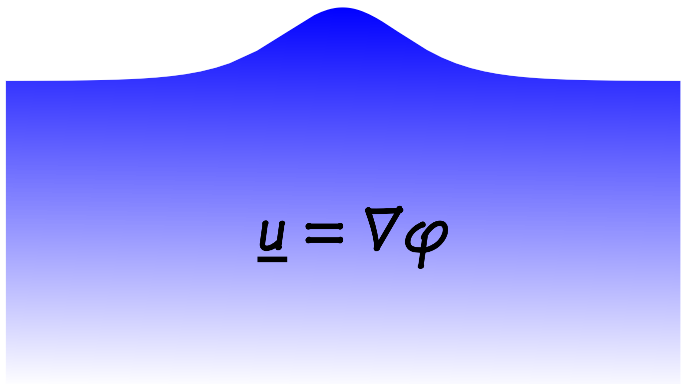
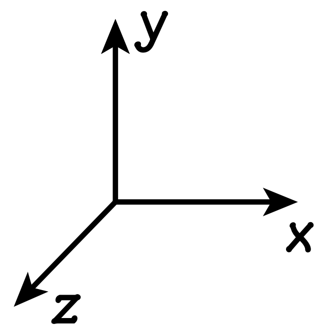
THE WATER-WAVE PROBLEM





Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + g\eta$$
$$-\sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \sigma\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

Difficulties:

- A free-boundary value problem
- Nonlinear boundary conditions

Solitary waves:

$$\eta(x, z, t) = \eta(x - ct, z), \;\; \varphi(x, y, z, t) = \varphi(x - ct, y, z)$$
$$\eta(x - ct, z) \to 0, \qquad |x - ct| \to \infty$$

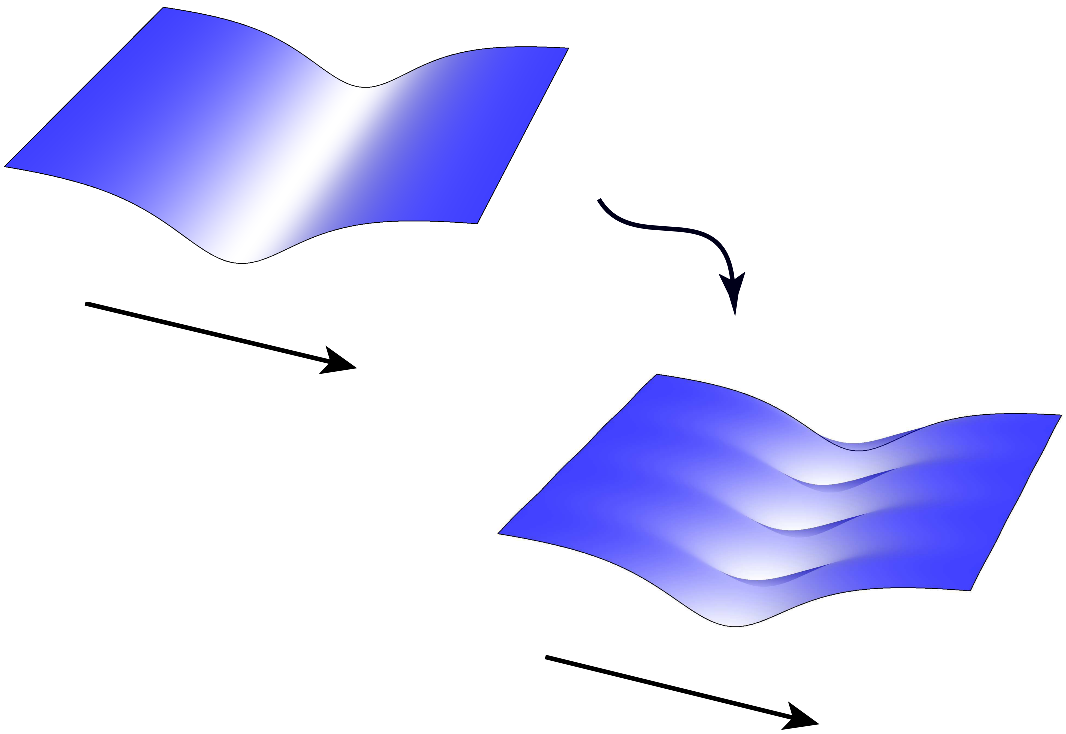
Parameter:

$$\alpha = gh/c^2, \quad \beta = \sigma/hc^2$$

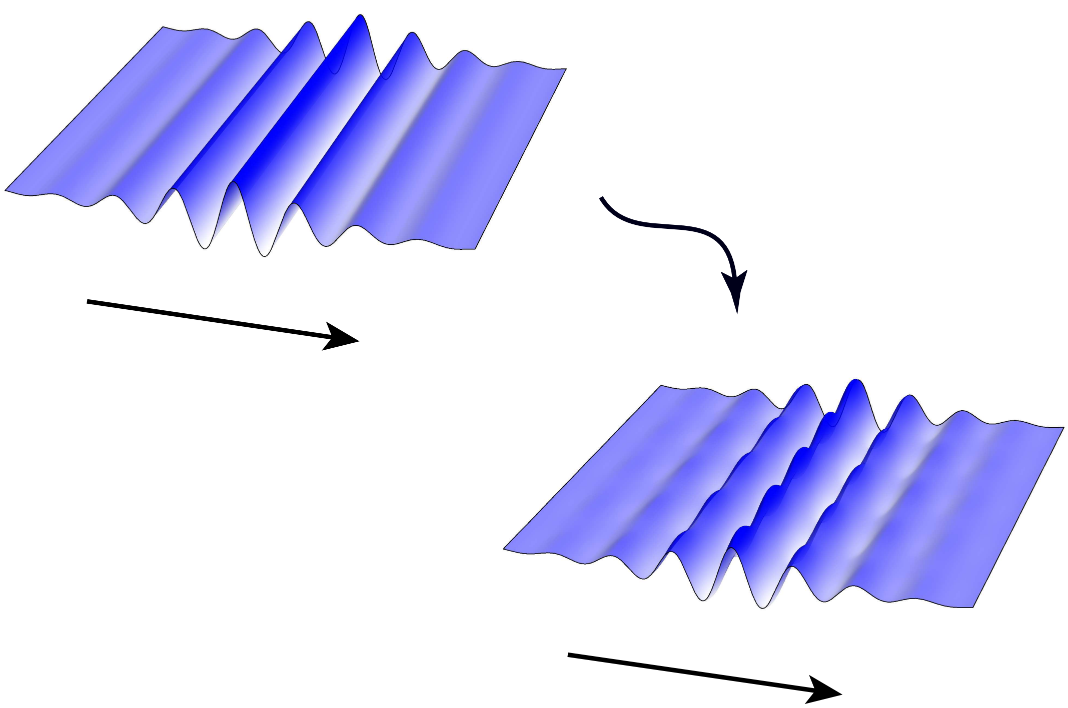
DIMENSION BREAKING

Periodically modulated solitary waves bifurcate from line solitary waves

- Strong surface tension ($\beta > 1/3$)



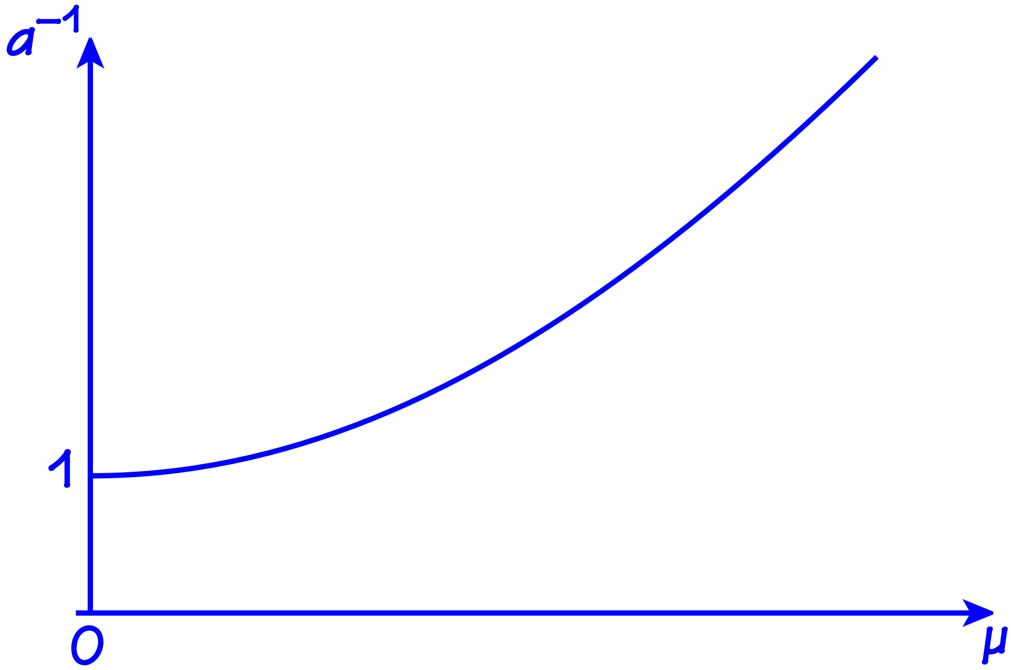
- Weak surface tension ($\beta < 1/3$)



MODELLING

Strong surface tension ($\beta > 1/3$):

- Dispersion relation:



- Write $a = 1 + \varepsilon^2$
- The Ansatz

$$\eta(x,z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z) + O(\varepsilon^4)$$

leads to the Kadomtsev-Petviashvili equation

$$\partial_{xx}\left(\zeta_{xx} - \zeta + \frac{3}{2}\zeta^2\right) - \zeta_{zz} = 0$$

- The KP equation admits line (KdV) and periodically modulated solitary-wave solutions

MODELLING

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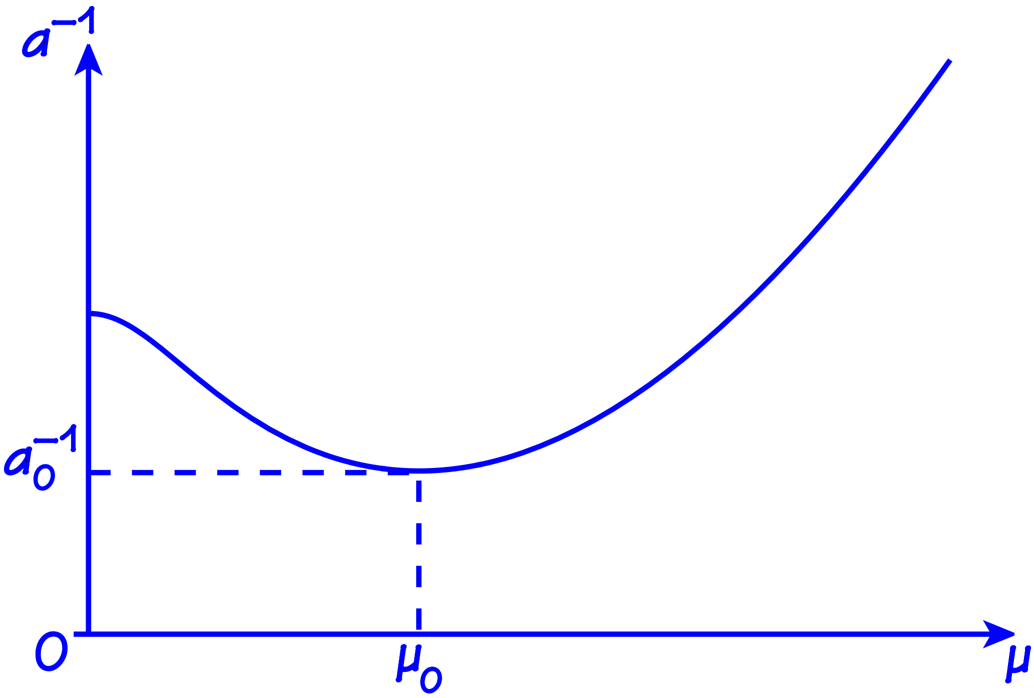
- Dispersion relation:



- Write $a = a_0 + \varepsilon^2$
- The Ansatz

$$\eta(x,z) = \varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{i\mu_0 x} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-i\mu_0 x}\right) + O(\varepsilon^2)$$

leads to the Davey-Stewartson system

$$\begin{aligned} \zeta - \zeta_{xx} - \zeta_{zz} - |\zeta|^2\zeta - \zeta\psi_x &= 0, \\ -\psi_{xx} - \psi_{zz} + (|\zeta|^2)_x &= 0 \end{aligned}$$

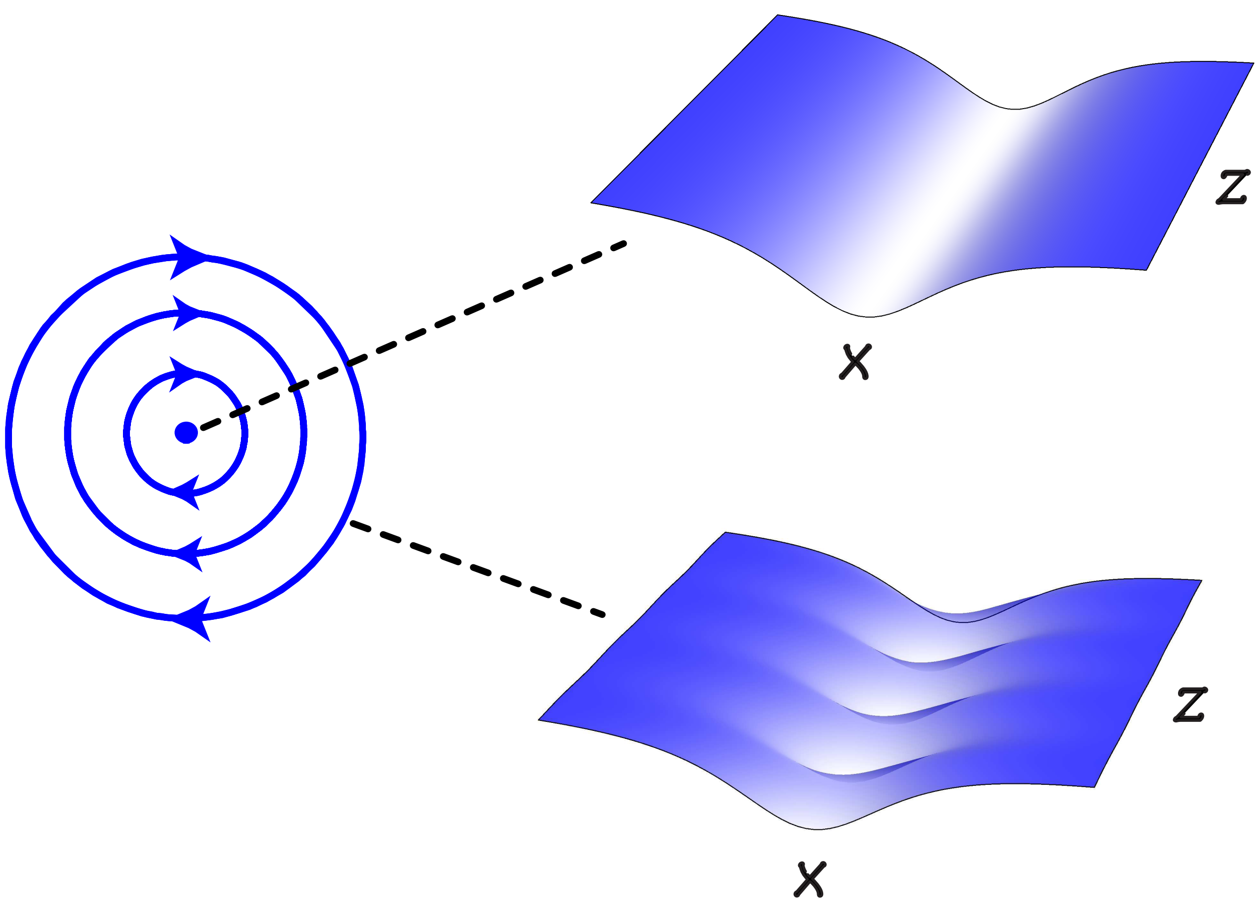
- The DS system has explicit line (NLS) and periodically modulated solitary-wave solutions

SPATIAL DYNAMICS

- Formulate the water-wave problem as an evolutionary equation

$$u_z = Lu + N(u), \qquad u \in X$$

  - z is the 'time-like' variable
  - X is a phase space of functions which vanish as $x \longrightarrow \pm\infty$
- Equilibrium solutions are line solitary waves
- Periodic solutions in the form of periodic orbits surrounding a nearby equilibrium are periodically modulated perturbations of the line solitary wave



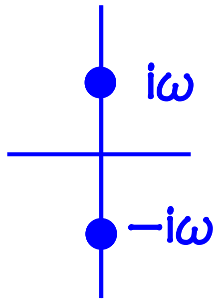
- Search for periodic orbits surrounding the equilibrium corresponding to the the KdV or NLS line solitary wave

LYAPUNOV CENTRE THEOREM

- Classical form for Hamiltonian systems

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \qquad j = 1, \dots, n$$

Linearise around an equilibrium $u^\star$



Nonresonance condition:

$in\omega$, $n \neq \pm 1$ is not an eigenvalue

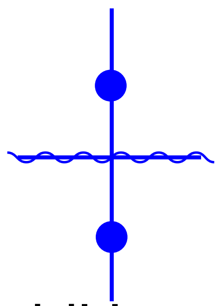
There exists a family $\{u_s\}$ of $2\pi/\omega_s$-periodic solutions with

$$u_s \longrightarrow u^\star, \quad \omega_s \longrightarrow \omega \qquad \text{as } s \longrightarrow 0$$

- Devaney extension: (infinite-dimensional) reversible systems

$$\dot{u} = Lu + N(u), \qquad S^2 = I, \quad SL = -LS, \quad SN = -NS$$

- Iooss extension:



Nonresonance condition violated:

$0 \in \sigma_{ess}(L)$

Additional condition:

$$Lu = N(u^\dagger)$$

is solvable for each $u^\dagger$

VARIATIONAL PRINCIPLE

- Luke's variational principle:

$$\delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_0^{1+\eta} \left(-\varphi_x + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \right) dy + \frac{1}{2}a\eta^2 + \beta\left(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1\right) \right\} dx\, dz = 0$$

- New variables:

$$\tilde{y} = y/(1 + \eta(x,z)), \qquad \varphi(x,y,z) = \Phi(x,\tilde{y},z)$$

$$\Rightarrow \qquad \delta\mathcal{L} = 0, \qquad \delta\mathcal{L} = \int_{-\infty}^{\infty} L(\eta, \Phi, \eta_z, \Phi_z)\, dz$$

- Legendre transform:

$$\omega = \frac{\delta L}{\delta \eta_z}, \qquad \xi = \frac{\delta L}{\delta \Phi_z}$$

$$\Rightarrow \quad \eta_z = \eta_z(\eta, \omega, \Phi, \xi), \qquad \Phi_z = \Phi_z(\eta, \omega, \Phi, \xi)$$

- Hamiltonian:

$$H(\eta, \omega, \Phi, \xi) = \int_{-\infty}^{\infty} \int_0^1 \Phi_z \xi\, d\tilde{y}\, dz + \int_{-\infty}^{\infty} \eta_z \omega\, dz - L(\eta, \Phi, \eta_z, \Phi_z)$$

- Hamilton's equations:

$$\eta_z = \frac{\delta H}{\delta \eta}, \quad \omega_z = -\frac{\delta H}{\delta \rho}, \quad \Phi_z = \frac{\delta H}{\delta \xi}, \quad \xi_z = -\frac{\delta H}{\delta \Phi}$$

(with boundary conditions for Φ at $y = 0, 1$)

- Reversibility: $(\eta, \omega, \Phi, \xi) \mapsto (\eta, -\omega, \Phi, -\xi)$

LINEAR SPECTRAL ANALYSIS

- Resolvent equations for $u = (\eta, \omega, \Phi, \xi)$:

$$(L - ik\varepsilon I)u = u^\dagger, \qquad 0 < k \leq k_{max}$$

- Solve for ω, ξ, Φ as functions of η, $u^\dagger$

$$\Rightarrow \qquad g(D)\eta = \mathcal{N}(\eta, u^\dagger),$$

where

$$g(\mu) = a_0 + \beta q^2 - \frac{\mu^2}{q}\coth q, \qquad q = \sqrt{\mu^2 + \varepsilon^2 k^2}$$

- $g(\mu) \geq 0$ with equality iff $\mu = \pm\mu_0$
- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1 - \chi(D))\eta$$

where χ is the indicator function of $[\pm\mu_0 - \delta, \pm\mu_0 + \delta]$
- Solve for η_2 as a function of η_1 and $u^\dagger$
- Write

$$\eta_1(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z)$$

or

$$\eta_1(x, z) = \varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{i\omega x} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-i\omega x}\right)$$

to arrive at the reduced system

$$\zeta_{xxxx} - \zeta_{xx} + 3(\zeta^\star\zeta_x)_x + 3(\zeta\zeta^\star_x)_x + O(\varepsilon) + k^2\zeta = \zeta^\dagger$$

or

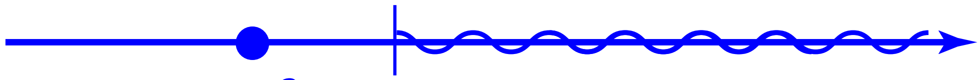
$$\begin{aligned} \zeta - \zeta_{xx} - 3(\zeta^\star)^2\zeta - (\zeta^\star)^2\overline{\zeta} - \zeta^\star\psi_x + O(\varepsilon) + k^2\zeta &= \zeta^\dagger, \\ -\psi_{xx} - 2(\mathrm{Re}\,\zeta^\star\zeta)_x + k^2\psi &= 0 \end{aligned}$$

LINEAR SPECTRAL ANALYSIS

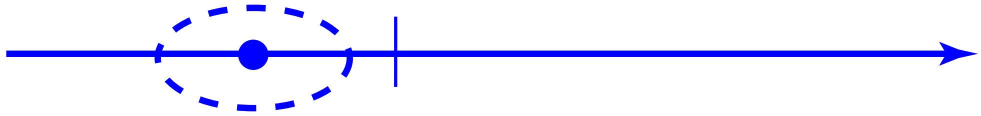
- Reduced equation:

$$(\mathcal{B}_{\varepsilon,k} + k^2 I)w = w^\dagger$$

  - $\mathcal{B}_{0,k}$ is known explicity, is self-adjoint and does not depend upon k
  - Spectrum of $\mathcal{B}_{0,k}$:



- Spectral perturbation for $k \in [k_{min}, k_{max}]$:



  - The point $-k^2$ lies inside the ellipse
  - $\mathcal{B}_{\varepsilon,k} + k^2 I$ is invertible if and only if $\lambda_{\varepsilon,k} + k^2 \neq 0$, otherwise it has a simple eigenvalue
  - $\lambda_{\varepsilon,k} + k^2 = 0$ has exactly one solution k_ε with $k_\varepsilon = \omega + O(\varepsilon)$

- $\pm i\varepsilon k_\varepsilon$ are simple eigenvalues of L and $L - i\lambda I$ is invertible for all other values of $|\lambda| > \varepsilon k_{min}$
- Iooss condition:
  - $Lu = N(u^\dagger)$ leads to $C_{\varepsilon,k}\zeta = \zeta^\dagger$
  - $C_{0,0}$ is known explicity and invertible
  - Hence $C_{\varepsilon,0}$ is invertible