THE KDV EQUATION

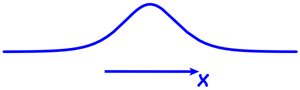
- KdV equation:

$$u_t + \left(1 + \tfrac{1}{6}u_{xx} + u^2\right)_x = 0$$

or

$$u_t + (m_{KdV}(D)u + u^2)_x = 0, \qquad m_{KdV}(k) = 1 - \tfrac{1}{6}k^2$$

- The KdV equation has solitary-wave solutions:



- Whitham equation (full dispersion KdV equation):

$$u_t + (m_{fKdV}(D)u + u^2)_x = 0, \qquad m_{fKdV}(k) = \left(\frac{\tanh(k)}{k}\right)^{\frac{1}{2}}$$

- The Whitham equation has small-amplitude solitary waves which are approximated by rescalings of KdV solitary waves (Ehrnström, Groves & Wahlén 2012).

THE KP-I EQUATION

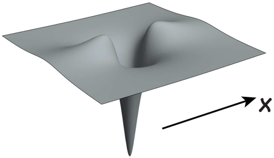
- KP-I equation:

$$u_t + \left((\beta - \tfrac{1}{3})u_{xx} - u + \tfrac{3}{2}u^2\right)_{xxx} - u_{xzz} = 0$$

or

$$u_t + (m_{KP}(D)u + \tfrac{3}{2}u^2)_x = 0, \qquad m_{KP}(k) = 1 + (\beta - \tfrac{1}{3})k_1^2 + \frac{k_2^2}{k_1^2}$$

- The KP-I equation has fully localised solitary-wave solutions:



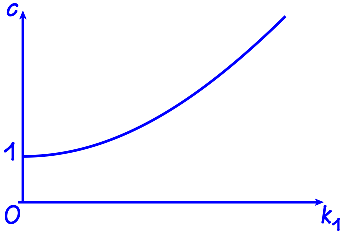
- Full dispersion KP-I equation:

$$u_t + (m_{fKP}(D)u + \tfrac{3}{2}u^2)_x = 0, \quad m_{fKP}(k) = \left((1 + \beta|k|^2)\frac{\tanh|k|}{|k|}\right)^{\frac{1}{2}}\left(1 + \frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}}$$

- Does the full dispersion KP-I equation have small-amplitude fully localised solitary waves which are approximated by rescalings of KP-I fully localised solitary waves?

FORMAL REDUCTION

- Dispersion relation for one-dimensional linear wave trains:



- The Ansatz

$$c = 1 - \varepsilon^2, \qquad u(x,z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z) + O(\varepsilon^4)$$

reduces the steady fKP-I equation

$$-cu + m_{\text{fKP}}(D) + \tfrac{3}{2}u^2 = 0$$

to the stationary KP-I equation

$$m_{\text{KP}}(D)\zeta + \tfrac{3}{2}\zeta^2 + O(\varepsilon) = 0$$

- m_{fKP} is an analytic function of k_1 and $\frac{k_2}{k_1}$ with

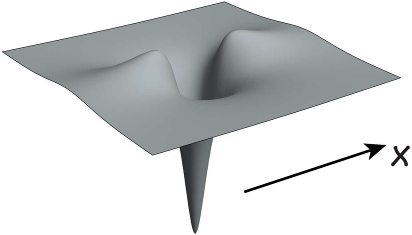
$$m_{\text{fKP}}(k) = m_{\text{KP}}(k) + O(|(k_1, \tfrac{k_2}{k_1})|^4)$$

STATIONARY KP-I EQUATION

$$m_{KP}(D)\zeta + \tfrac{3}{2}\zeta^2 = 0$$

- An explicit solitary-wave solution

$$\zeta(x,z) = -8\frac{3-(x^2-z^2)/(\beta-\frac{1}{3})}{(3+(z^2+z^2)/(\beta-\frac{1}{3}))^2}$$



- This solution is a critical point of the functional

$$\tilde{I}_0(\zeta) = \frac{1}{2}\int_{\mathbb{R}^2} m_{KP}(k)|\hat{\zeta}|^2\,dk_1\,dk_2 - \frac{1}{3}\int_{\mathbb{R}^2}\zeta^3\,dx\,dz$$

with function space

$$X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

A VARIATIONAL PRINCIPLE

Fully localised solitary-wave solutions of the full-dispersion KP-I equation

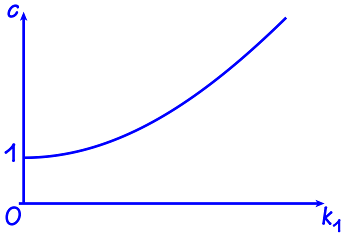
$$u_t + (m_{\mathrm{fKP}}(D)u + \tfrac{3}{2}u^2)_x = 0,$$

are critical points of the functional

$$J(u) = -\frac{1}{2}c\int_{\mathbb{R}^2} u^2\,dx\,dz + \frac{1}{2}\int_{\mathbb{R}^2} m_{\mathrm{fKP}}(k)|\hat{u}|^2\,dk_1\,dk_2 - \frac{1}{3}\int_{\mathbb{R}^2} u^3\,dx\,dz$$

REDUCTION

- Modelling:



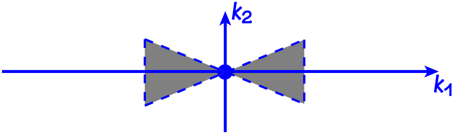
$$c = 1 - \varepsilon^2,$$

$$u(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z) + O(\varepsilon^4)$$

- Write

$$u_1 = \chi(D)u, \qquad u_2 = (1 - \chi(D))u,$$

where χ is the characteristic function of this set:



$$|k_1| < \delta, \left|\frac{k_2}{k_1}\right| < \delta$$

REDUCTION

- $J'(u) = 0 \qquad \Rightarrow \qquad \begin{aligned} &\chi(D)J'(u_1 + u_2) = 0, \\ &(1 - \chi(D))J'(u_1 + u_2) = 0 \end{aligned}$

  Solve for $u_2 = u_2(u_1)$, set $\mathcal{J}(u_1) = J(u_1 + u_2(u_1))$, consider $\mathcal{J}'(u_1) = 0$
- Write

$$u_1(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z)$$

- Arrive at the reduced variational functional

$$\mathcal{T}_\varepsilon(\zeta) = \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} m_{KP}(k)|\hat{\zeta}|^2 \, dk_1 \, dk_2 - \frac{1}{3}\int_{\mathbb{R}^2} \zeta^3 \, dx \, dz}_{= \mathcal{T}_0(\zeta)} + O(\varepsilon^{1/2}\|\zeta\|^2)$$

- Study this functional in

$$B_R(0) \subseteq X_\varepsilon := \chi(\varepsilon D_1, \varepsilon^2 D_2)X, \qquad X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

NATURAL CONSTRAINT SET

Find critical points of

$$\tilde{I}_0(\zeta) = \frac{1}{2}\|\zeta\|^2 - K(\eta), \qquad K(\eta) = \frac{1}{3}\int_{\mathbb{R}^2} \zeta^3 \, dx \, dz$$

using the natural constraint set

$$N := \{\zeta \neq 0 : \langle \tilde{I}_0'(\zeta), \zeta \rangle = 0\}$$

- Every critical point of $\tilde{I}_0$ lies on N
- Any critical point of $\zeta^\star$ of $\tilde{I}_0|_N$ is a critical point of $\tilde{I}_0$:
  - Set $F(\zeta) = \langle \tilde{I}_0'(\zeta), \zeta \rangle$
  - There is a Lagrange multiplier μ with $\tilde{I}_0'(\zeta^\star) - \mu F'(\zeta^\star) = 0$
  - However

$$\mu = -\frac{\langle \tilde{I}_0'(\zeta^\star) - \mu F'(\zeta^\star), \zeta^\star \rangle}{\langle F'(\zeta^\star), \zeta^\star \rangle} = 0$$

  because

$$\langle F'(\zeta), \zeta \rangle = -3\|\zeta\|^2 < 0, \qquad \zeta \in N$$

- Look for minimisers of $\tilde{I}_0$ over N

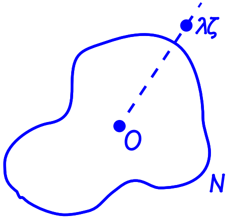
GEOMETRICAL INTERPRETATION

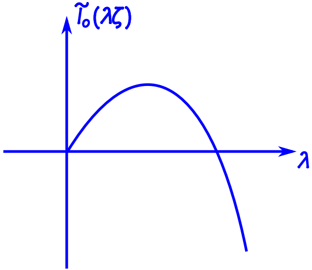
$$N = \{\zeta \neq 0 : \langle \tilde{T}_0'(\zeta), \zeta \rangle = 0\}$$

- Any point $\zeta \in N$ satisfies $K(\zeta) > 0$
- Any ray

$$\{\lambda\zeta : K(\zeta) > 0, \lambda > 0\}$$

intersects N in precisely one point and the value of $\tilde{T}_0$ along such a ray attains a strict maximum at this point (examine $\tilde{T}_0(\lambda\zeta)$)





- $\tilde{T}_0(\zeta)$ (and $K(\zeta)$, $||\zeta||$) are bounded below above zero on N

EXISTENCE THEORY

How to find a minimiser for $\tilde{\mathcal{I}}_0(\zeta) = \frac{1}{2}\|\zeta\|^2 - \mathcal{K}(\eta)$ over

$$N = \{\zeta \neq 0 : \underbrace{\langle \tilde{\mathcal{I}}_0'(\zeta), \zeta \rangle}_{:= F(\zeta)} = 0\}?$$

- Lemma (Palais-Smale sequence):
  There exists a minimising sequence $\{\zeta_n\}$ for $\tilde{\mathcal{I}}_0|_N$ with $\tilde{\mathcal{I}}_0'(\zeta_n) \to 0$

  – Take a minimising sequence $\{\zeta_n\}$ for $\tilde{\mathcal{I}}_0|_N$

  – By Ekeland's variational principle there exists a sequence of real numbers with $\tilde{\mathcal{I}}_0'(\zeta_n) - \mu_n F'(\zeta_n) \to 0$

  – Our previous argument shows that $\mu_n \to 0$

- Theorem (concentration-compactness):
  There is a sequence $\{w_n\} \subset \mathbb{R}^2$ such that that a subsequence of $\{\zeta_n(\cdot + w_n)\}$ converges weakly to a minimiser ζ_∞ of $\tilde{\mathcal{I}}_0|_N$

  – Vanishing leads to $\mathcal{K}(\zeta_n) \to 0$

  – Dichotomy leads to n critical points with $\tilde{\mathcal{I}}_0(\zeta^1) + \cdots + \tilde{\mathcal{I}}_0(\zeta^n) = \inf \tilde{\mathcal{I}}_0|_N$