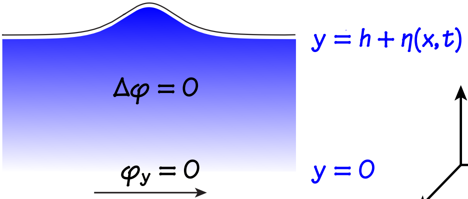
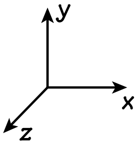
HYDRODYNAMIC PROBLEM





Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x \varphi_x$$

Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta$$

$$+\frac{D}{\rho}\left(\frac{1}{(1+\eta_x^2)^{1/2}}\left[\frac{1}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3\right) = 0$$

Difficulties:

- A free-boundary value problem
- Nonlinear boundary conditions

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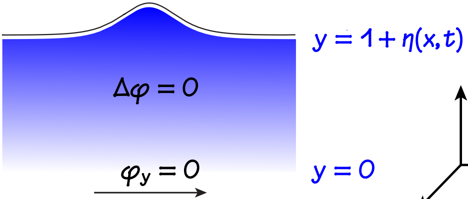
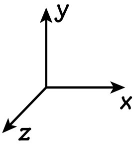




Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x \varphi_x$$

Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + \eta + \gamma\left(\frac{1}{(1+\eta_x^2)^{1/2}}\left[\frac{1}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3\right) = 0$$

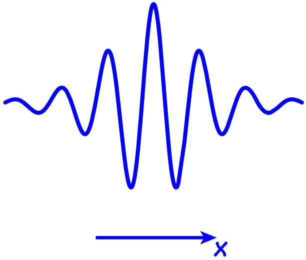
Difficulties:

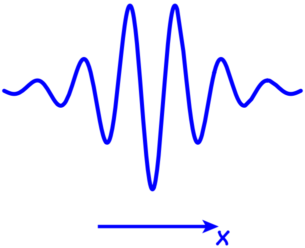
- A free-boundary value problem
- Nonlinear boundary conditions

Parameter: $\gamma = D/\rho g h^4$

SOLITARY WAVES

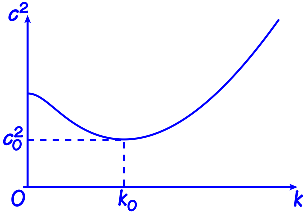
- $\eta(x,t) = \eta(x - ct)$
- $\eta(x - ct) \longrightarrow 0$ as $x - ct \longrightarrow \pm\infty$





MODELLING

- Dispersion relation for periodic wave trains ($\eta \sim \cos kx$):



- The Ansatz

$$c^2 = c_0^2(1-\mu^2), \qquad \eta(x) = \mu\left(\zeta(\mu x)e^{ik_0x} + \overline{\zeta(\mu x)}e^{-ik_0x}\right) + O(\mu^2)$$

leads to the nonlinear Schrödinger equation

$$\zeta_{xx} - \zeta \pm |\zeta|^2\zeta = 0$$

- ‘Focussing’ (+) – with solitary waves $e^{i\omega x}\zeta_{NLS}(x+x_0)$ – for $\gamma > \gamma_0 \geq 3.37 \times 10^{-10}$
- Typical values:
  - $\gamma \sim 10^{-5}$ (McMurdo sound)
  - $\gamma \sim 10^{-2}$ (Lake Saroma)

VARIATIONAL PRINCIPLE

Minimise the energy

$$\mathcal{H}(\eta,\varphi)=\int_{-\infty}^{\infty}\left\{\int_{0}^{1+\eta}\left(\frac{1}{2}(\varphi_x^2+\varphi_y^2)\right)dy+\frac{1}{2}\eta^2+\gamma\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}\right\}dx$$

subject to fixed momentum

$$I(\eta,\phi)=\int_{-\infty}^{\infty}\eta_x\varphi|_{y=\eta}\,dx=2c_0\mu,\qquad 0<\mu\ll 1;$$

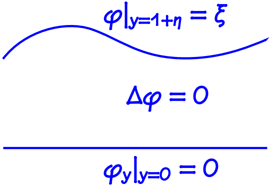
the Lagrange multiplier is the wave speed.

- *H* and *I* are conserved quantities
- Yields *conditional, energetic* stability of the *set* of minimisers

DIRICHLET-NEUMANN OPERATOR

- Use a Dirichlet-Neumann operator:

$$G(\eta)\xi = \sqrt{1+\eta_x^2}\,\varphi_n|_{y=1+\eta}$$



- Minimise

$$\mathcal{H}(\eta,\xi) = \int_{-\infty}^{\infty}\left\{\frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \gamma\frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}\right\}dx$$

subject to fixed

$$I(\eta,\xi) = \int_{-\infty}^{\infty}\eta_x\xi\,dx = 2c_0\mu,$$

where $\xi = \phi|_{y=1+\eta}$

REFORMULATION

Minimise

$$\mathcal{H}(\eta,\xi) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \gamma \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\}$$

subject to fixed

$$I(\eta,\xi) = \int_{-\infty}^{\infty} \eta_x \xi \, dx = 2c_0\mu$$

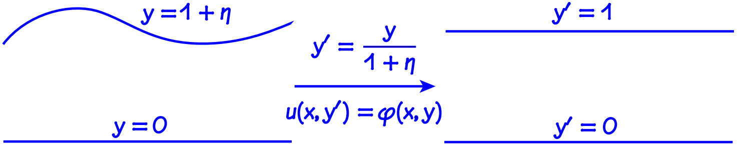
- Fix η and minimise $H(\eta,\xi)$ over $I(\eta,\xi) = 2c_0\mu$. There is a unique minimiser ξ_η with
$$G(\eta)\xi_\eta = c_\eta \eta_x$$
- Minimise
$$J(\eta) = H(\eta,\xi_\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)},$$
where
$$\mathcal{K}(\eta) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\eta^2 + \gamma \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\}, \qquad \mathcal{L}(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \eta_x G(\eta)^{-1}\eta_x$$
- We show that
  - $J(\eta)$ has a minimiser
  - Minimising sequences converge (up to subsequences/translations)

ANALYTICITY

$$K(\eta)\xi = -\partial_x(G(\eta)^{-1}\xi_x)$$

- $(\eta, \xi) \mapsto K(\eta)\xi$ is analytic at the origin

- Flatten the domain:



$$K(\eta)\xi = -(\varphi|_{y=\eta})_x \qquad K(\eta)\xi = -u_x|_{y'=1}$$

$$\begin{aligned} \Delta\varphi &= 0, & \Delta' u - \partial_x F_1(\eta, u) - \partial_{y'} F_2(\eta, u) &= 0, \\ \varphi_y - \eta_x \varphi_x - \xi_x &= 0, & u_{y'} - F_2(\eta, u) - \xi_x &= 0, \\ \varphi_y &= 0, & u_{y'} &= 0 \end{aligned}$$

ANALYTICITY

$$\left.\begin{array}{r} \Delta' u - \partial_x F_1(\eta, u) - \partial_{y'} F_2(\eta, u) \\ u_{y'} - F_2(\eta, u) - \xi_x|_{y'=1} \\ u_{y'}|_{y'=0} \end{array}\right\} = 0$$

- The left-hand side is an analytic function of u, η and ξ
- $(u, \eta, \xi) = (0, 0, 0)$ is a solution
- The linearisation of the left-hand side with respect to u at $(\eta, \xi) = (0, 0)$ is invertible:

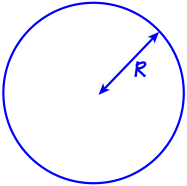
$$\begin{aligned} \Delta' u &= G \\ u_{y'}|_{y'=1} &= g_1, \\ u_{y'}|_{y'=0} &= g_0 \end{aligned}$$

  has a unique solution for each G, g_1, g_0
- By the analytic implicit function theorem u is an analytic function of (η, ξ) at the origin.
- Hence $(\eta, \xi) \mapsto K(\eta)\xi$ is also analytic at the origin (since $K(\eta)\xi = -u_x|_{y'=1}$)

MINIMISATION PROCEDURE

Pretend $\mathbb{R}$ is bounded!

- Work in a ball of radius R in $H^2(\mathbb{R})$



$$\|\eta\|_2 = \left(\int (\eta^2 + \eta_{xx}^2)\right)^{1/2}$$

- A standard argument shows that

$$J(\eta) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\eta^2 + \gamma \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} + \frac{\mu^2}{\mathcal{L}(\eta)}$$

  has a minimiser η_{min} over $\overline{B}_R(0)$

- η_{min} is obviously not zero
- How to show that η_{min} is not on the boundary of $\overline{B}_R(0)$?
  - $\|\eta\|_2^2 \lesssim J(\eta)$ in $\overline{B}_R(0)$
  - The test function

    $$\eta^*(x) = \mu\zeta_{NLS}(\mu x)\cos k_0 x - A\mu^2\zeta_{NLS}(\mu x)^2\cos(2k_0 x) - B\mu^2\zeta_{NLS}(\mu x)^2$$

    satisfies $J(\eta^*) < 2c_0\mu$

SOLITARY WAVES

- Use *concentration-compactness* – minimising sequences undergo *concentration*, *vanishing* or *dichotomy*
- Rule out dichotomy by showing that $i_\mu = \inf J$ satisfies $i_{\mu_1+\mu_2} < i_{\mu_1} + i_{\mu_2}$ ('strict sub-additivity')

  Difficulties:

  – Nonlocal equations
  – Inhomogeneous nonlinearities
- Our equations are 'almost' local
- We show that the functions in a minimising sequence $\{\eta_n\}$ 'scale' like the test function

$$\eta^\star(x) = \mu\zeta_{\mathrm{NLS}}(\mu x)\cos k_0 x - A\mu^2\zeta_{\mathrm{NLS}}(\mu x)^2\cos(2k_0 x) - B\mu^2\zeta_{\mathrm{NLS}}(\mu x)^2$$

for which

$$J(\eta^\star) = 2c_0\mu - C\mu^3\int(\eta^\star)^4 + o(\mu^3)$$

SOLITARY WAVES

- $\mathcal{J}'(\eta_n) \to 0$ and $\|\eta_n\|_2^2 \lesssim \mu$
- Write

$$\eta_{n,1} = \chi(D)\eta_n, \qquad \eta_{n,2} = (1-\chi(D))\eta_n,$$

where χ is the characteristic function of this set:

$$|k - k_0| < \delta$$

- Variational reduction:

$$\mathcal{J}'(\eta_n) = 0 \qquad \Rightarrow \qquad \begin{array}{l} \chi(D)\mathcal{J}'(\eta_{n,1}+\eta_{n,2}) = 0, \\ (1-\chi(D))\mathcal{J}'(\eta_{n,1}+\eta_{n,2}) = 0 \end{array}$$

Solve for $\eta_{2,n} = \eta_{2,n}(\eta_{n,1})$, set $\tilde{\mathcal{J}}(\eta_{n,1}) = \mathcal{J}(\eta_{n,1} + \eta_{2,n}(\eta_{n,1}))$ and consider $\tilde{\mathcal{J}}'(\eta_{n,1}) = 0$

- Write

$$\eta_{1.n}(x) = \tfrac{1}{2}\mu\zeta_n(\mu x)e^{ik_0x} + \tfrac{1}{2}\mu\overline{\zeta_n(\mu x)}e^{-ik_0x}$$

and show that $\|\zeta_n\|^2 \lesssim \mu$, $\|\eta_{2,n}\|^2 \lesssim \mu^3$.

CONVERGENCE TO NLS

- The set D_μ of minimisers of J_μ satisfies

$$\sup_{\eta \in D_\mu} \inf_{\omega \in [0,2\pi], x_0 \in \mathbb{R}} \|\zeta_\eta - e^{i\omega}\zeta_{\mathrm{NLS}}(\cdot + x_0)\|_1 \to 0$$

as $\mu \downarrow 0$, where we write

$$\eta_1(x) = \tfrac{1}{2}\mu\zeta_\eta(\mu x)e^{ik_0x} + \tfrac{1}{2}\mu\overline{\zeta_\eta(\mu x)}e^{-ik_0x}$$

- Furthermore, the wave speed for a minimiser η satisfies

$$c_\eta = c_0 - C_{\mathrm{NLS}}\mu^2 + o(\mu^2)$$

uniformly over D_μ.