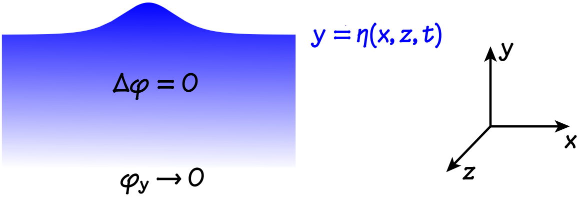
THE WATER-WAVE PROBLEM



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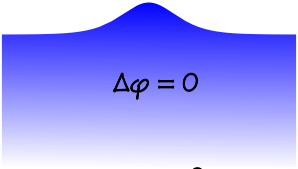

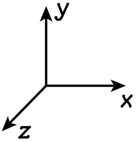

Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

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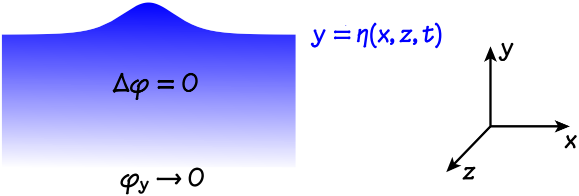

Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \sigma\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

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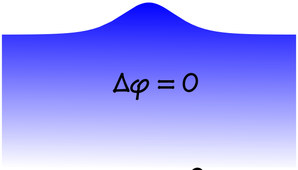
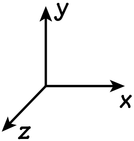


$y = \eta(x, z, t)$

$\varphi_y \longrightarrow 0$



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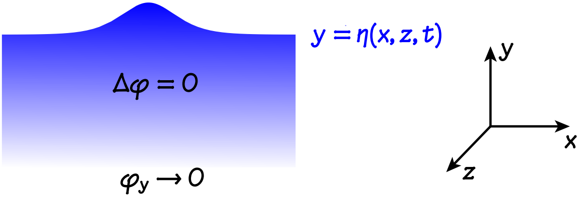
Difficulties:

THE WATER-WAVE PROBLEM



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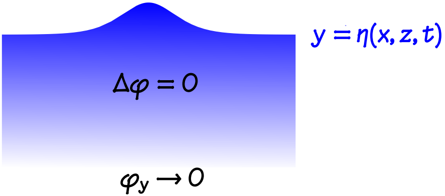
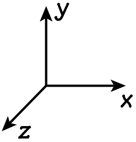
Difficulties:

- A free-boundary value problem

THE WATER-WAVE PROBLEM





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Difficulties:

- A free-boundary value problem
- Nonlinear boundary conditions

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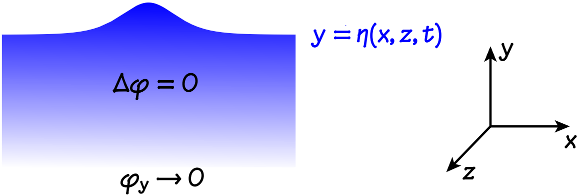

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Dynamical boundary condition:

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Difficulties:

- A free-boundary value problem
- Nonlinear boundary conditions

Nondimensionalise

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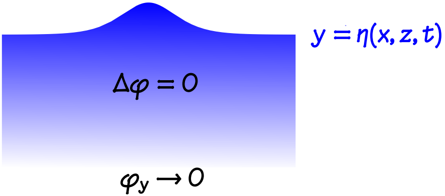
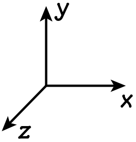




Kinematic boundary condition:

$$-c\eta_x = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

Dynamical boundary condition:

$$-c\varphi_x + \frac{1}{2}|\nabla\varphi|^2 + \eta - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

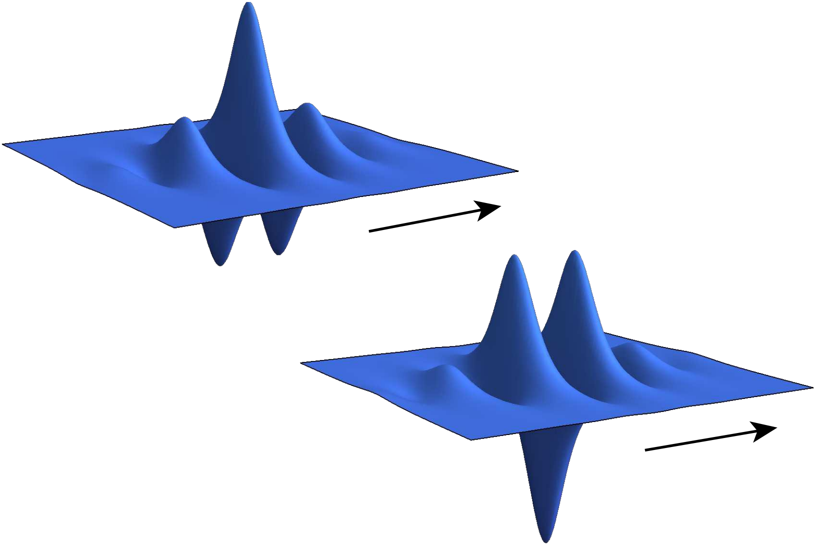
Difficulties:

- A free-boundary value problem
- Nonlinear boundary conditions

Nondimensionalise

Solitary waves: $\eta(x, z, t) = \eta(x - ct, z)$, $\eta(x - ct, z) \to 0$ as $|(x - ct, z)| \to \infty$

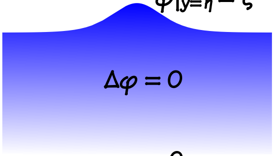
FULLY LOCALISED SOLITARY WAVES



FORMULATION

- Use a Dirichlet-Neumann operator:

$$\varphi|_{y=\eta} = \xi$$



$$\Delta\varphi = 0$$

$$\varphi_y \to 0$$

$$G(\eta)\xi = \sqrt{1 + \eta_x^2 + \eta_z^2}\, \varphi_n|_{y=\eta}$$

FORMULATION

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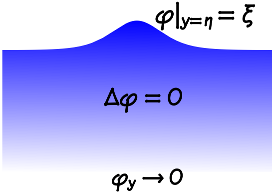

$$G(\eta)\xi = \sqrt{1 + \eta_x^2 + \eta_z^2}\,\varphi_n|_{y=\eta}$$

- We obtain the Zakharov-Craig-Sulem formulation

$$\eta_t - G(\eta)\xi = 0,$$

$$\xi_t + \eta + \frac{1}{2}\xi_x^2 + \frac{1}{2}\xi_z^2 - \frac{(G(\eta)\xi + \eta_x\xi_x + \eta_z\xi_z)^2}{2(1 + \eta_x^2 + \eta_z^2)}$$

$$- \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z = 0$$

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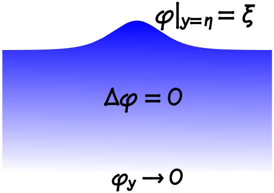

$$G(\eta)\xi = \sqrt{1+\eta_x^2+\eta_z^2}\,\varphi_n|_{y=\eta}$$

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$$-c\eta_x - G(\eta)\xi = 0,$$

$$-c\xi_x + \eta + \frac{1}{2}\xi_x^2 + \frac{1}{2}\xi_z^2 - \frac{(G(\eta)\xi + \eta_x\xi_x + \eta_z\xi_z)^2}{2(1+\eta_x^2+\eta_z^2)} - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

- Travelling waves $\eta(x,z,t) = \eta(x-ct,z)$, $\xi(x,z,t) = \xi(x-ct,z)$

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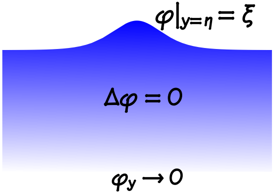

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- Travelling waves $\eta(x,z,t) = \eta(x-ct,z)$, $\xi(x,z,t) = \xi(x-ct,z)$
- Eliminate ξ using $\xi = -cG(\eta)^{-1}\eta_x$

FORMULATION

We arrive at the single equation

$$\mathcal{K}(\eta) - c^2\mathcal{L}(\eta) = 0,$$

where

$$\mathcal{K}(\eta) = \eta - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z,$$

$$\mathcal{L}(\eta) = -\frac{1}{2}(K(\eta)\eta)^2 - \frac{1}{2}(L(\eta)\eta)^2 + \frac{(\eta_x - \eta_x K(\eta)\eta - \eta_z L(\eta)\eta)^2}{2(1+\eta_x^2+\eta_z^2)} + K(\eta)\eta$$

and

$$K(\eta)\xi = -(G(\eta)^{-1}\xi_x)_x, \qquad L(\eta)\xi = -(G(\eta)^{-1}\xi_x)_z$$

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$$K(\eta)\xi = -(G(\eta)^{-1}\xi_x)_x, \qquad L(\eta)\xi = -(G(\eta)^{-1}\xi_x)_z$$

- Lemma: $K(\cdot)(\cdot)$, $L(\cdot)(\cdot)$: $H^3(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2) \to H^{3/2}(\mathbb{R}^2)$ are analytic at the origin.

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- Lemma: $K(\cdot)(\cdot), L(\cdot)(\cdot) \colon H^3(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2) \to H^{3/2}(\mathbb{R}^2)$ are analytic at the origin.
- Corollary: The functions $\mathcal{K}, \mathcal{L} : H^3(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$ are analytic at the origin.

ANALYTICITY

$$K(\eta)\xi = -(\varphi|_{y=\eta})_x, \qquad L(\eta)\xi = -(\varphi|_{y=\eta})_z,$$

where φ is the solution of the boundary-value problem

$$\begin{aligned}
&\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, && y < \eta, \\
&\varphi_y \to 0, && y \to -\infty, \\
&\varphi_y - \eta_x\varphi_x - \eta_z\varphi_z = \xi_x, && y = \eta
\end{aligned}$$

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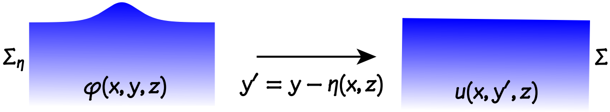
- Flatten:



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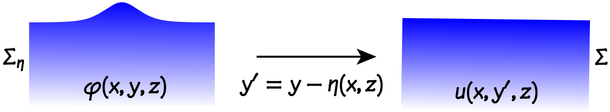

- New problem:

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} = \partial_x F_1(\eta, u) + \partial_y F_2(\eta, u) + \partial_z F_3(\eta, u), && y < 0, \\ &u_y \longrightarrow 0, && y \longrightarrow -\infty, \\ &u_y = F_2(\eta, u) + \xi_x, && y = 0, \end{aligned}$$

with $F_1(\eta, u) = \eta_x u_y$, $F_2(\eta, u) = \eta_x u_x + \eta_z u_z - (\eta_x^2 + \eta_z^2)u_y$, $F_3(\eta, u) = \eta_z u_y$

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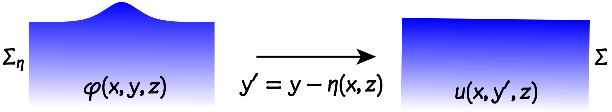

- New problem:

$$u_{xx} + u_{yy} + u_{zz} = \partial_x F_1(\eta, u) + \partial_y F_2(\eta, u) + \partial_z F_3(\eta, u), \qquad y < 0,$$
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- $K(\eta)\xi = -u_x|_{y=0}$, $L(\eta)\xi = -u_z|_{y=0}$

ANALYTICITY

- For each $F_1, F_2, F_3 \in H^2(\Sigma)$, $\xi \in H^{5/2}(\mathbb{R}^2)$ the BVP

$$
\begin{aligned}
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\end{aligned}
$$

admits a unique solution $u = \mathcal{S}(F_1, F_2, F_3, \xi)$ in $H^3_\star(\Sigma)$

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- Seek zeros of

$$T : H^3_\star(\Sigma) \times H^3(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2) \longrightarrow H^3_\star(\Sigma),$$

$$T(u, \eta, \xi) = u - \mathcal{S}(F_1(\eta, u), F_2(\eta, u), F_3(\eta, u), \xi)$$

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- $(\eta, u) \mapsto F_j(\eta, u)$ is analytic $H^3(\mathbb{R}^2) \times H^3_\star(\Sigma) \to H^2(\Sigma)$

ANALYTICITY

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- $\mathcal{T}$ is analytic with $\mathcal{T}(0, 0, 0) = 0$ and $d_1\mathcal{T}[0, 0, 0] = I$

ANALYTICITY

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- T is analytic with $T(0,0,0) = 0$ and $d_1 T[0,0,0] = I$
- Analytic implicit-function theorem: $u = u(\eta, \xi)$ is analytic $H^3(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2) \to H^3_\star(\Sigma)$

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- $K(\eta)\xi = -u_x|_{y=0}$, $L(\eta)\xi = -u_z|_{y=0}$ are analytic $H^3(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2) \longrightarrow H^{3/2}(\mathbb{R}^2)$

MODELLING

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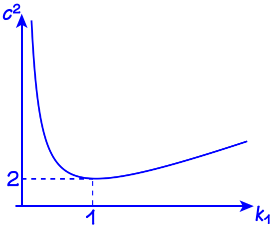

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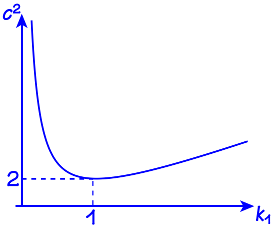

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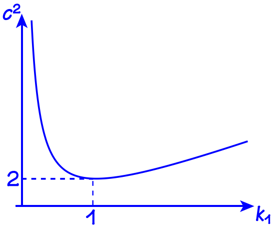

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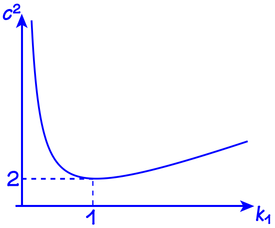

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where $X = \varepsilon x$, $Z = \varepsilon z$, into $(\star)$

- At $O(\varepsilon^3)$ one finds that

$$-\tfrac{1}{2}\zeta_{1xx} - \zeta_{1zz} + \zeta_1 - \tfrac{11}{16}|\zeta_1|^2\zeta_1 = 0$$

NLS EQUATION

- The stationary nonlinear Schrödinger equation

$$-\frac{1}{2}\zeta_{xx} - \zeta_{zz} + \zeta - \frac{11}{16}|\zeta|^2\zeta = 0$$

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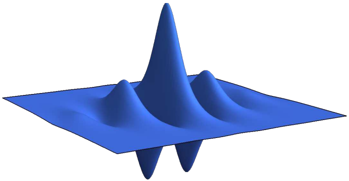
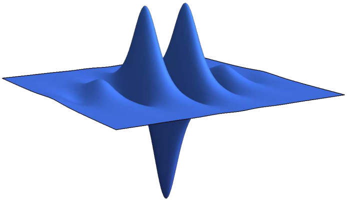
- Two of them ($\theta = 0$ and $\theta = \pi$) are symmetric:





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$$\mathcal{K}_1(\eta) = \eta - \eta_{xx} - \eta_{zz},$$

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- Writing

$$K(\eta)\xi = K_0\xi + K_1(\eta)\xi + K_2(\eta)\xi + \ldots, \quad L(\eta)\xi = L_0\xi + L_1(\eta)\xi + L_2(\eta)\xi + \ldots$$

yields

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Seek a solution η of

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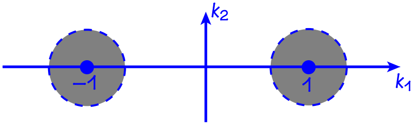
- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1-\chi(D))\eta,$$

where χ is the characteristic function of this set:



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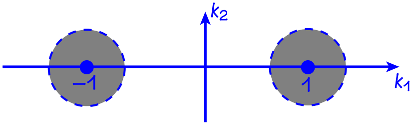

- Consider

$$\chi(D)\left(\mathcal{K}(\eta_1+\eta_2) - 2(1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2)\right) = 0, \qquad (1)$$

$$(1-\chi(D))\left(\mathcal{K}(\eta_1+\eta_2) - 2(1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2)\right) = 0 \qquad (2)$$

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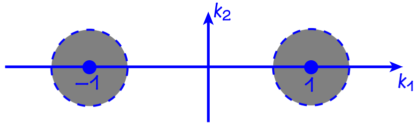

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- Solve (2) for $\eta_2 = \eta_2(\eta_1)$ and insert into (1) to yield the reduced equation

$$\chi(D)\left(\mathcal{K}(\eta_1+\eta_2(\eta_1)) - 2(1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2(\eta_1))\right) = 0$$

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where

$$g(k) = 1 + |k|^2 - 2\frac{k_1^2}{|k|},$$

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- In the limit $\varepsilon \to 0$ we obtain the stationary NLS equation

$$-\tfrac{1}{2}\zeta_{xx} - \zeta_{zz} + \zeta - \tfrac{11}{16}|\zeta|^2\zeta = 0$$

  from $(\star)$

EXISTENCE

- The two symmetric solutions $\pm\zeta^\star$ of

$$-\frac{1}{2}\zeta_{xx} - \zeta_{zz} + \zeta - \frac{11}{16}|\zeta|^2\zeta = 0$$

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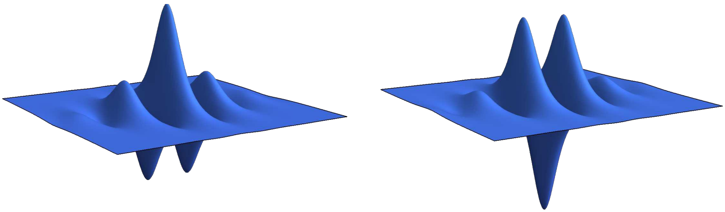
- $(\star)$ has two symmetric solitary waves for small values of ε



‘SMALL’ SOLUTIONS

- We have found solitary waves of the form

$$\eta = \eta_1 + \eta_2, \qquad \eta_1(x,z) = \tfrac{1}{2}\varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{ix} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-ix}\right)$$

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