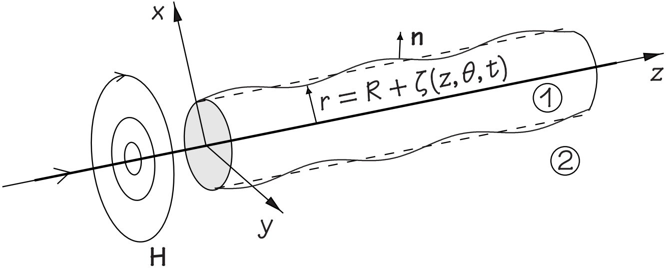
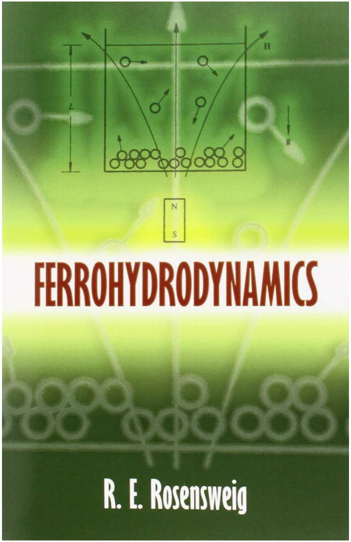


- Steady state:

$$\mathbf{H} = \frac{J}{2\pi r}\mathbf{e}_\theta, \qquad \mathbf{v} = \mathbf{0}, \qquad \zeta = 0$$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FERROHYDRODYNAMICS
R. E. Rosensweig

MAGNETIC EQUATIONS

- The magnetic field B, H satisfies

$$\operatorname{curl}\mathbf{H}_1 = \mathbf{0}, \qquad \operatorname{curl}\mathbf{H}_2 = \mathbf{0}, \qquad \operatorname{div}\mathbf{B}_1 = 0, \qquad \operatorname{div}\mathbf{B}_2 = 0,$$

where

$$\mathbf{B}_1 = \mu_0(\mathbf{H}_1 + \mathbf{M}(|\mathbf{H}_1|)), \qquad \mathbf{B}_2 = \mu_0\mathbf{H}_2$$

- Hence

$$\mathbf{H}_1 = -\nabla\psi_1, \qquad \mathbf{H}_2 = -\nabla\psi_2,$$

where

$$\operatorname{div}(\mu(|\nabla\psi_1|)\nabla\psi_1) = 0, \qquad \Delta\psi_2 = 0,$$

and

$$\mu(s) = 1 + \frac{|\mathbf{M}(s)|}{s}$$

- At the free surface $\mathrm{B}_1.\mathbf{n} = \mathrm{B}_2.\mathbf{n}$ and $\mathrm{H}_1.\mathbf{t} = \mathrm{H}_2.\mathbf{t}$, so that

$$\psi_2 = \psi_1, \qquad \psi_{2n} = \mu(|\nabla\psi_1|)\psi_{1n}$$

HYDRODYNAMIC EQUATIONS

- The velocity field $\mathbf{v}$ satisfies

$$\operatorname{curl} \mathbf{v} = \mathbf{0}, \qquad \operatorname{div} \mathbf{v} = \mathbf{0},$$

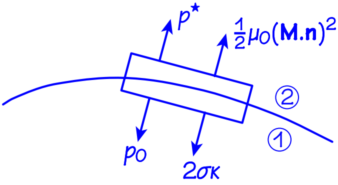
$$\rho \mathbf{v}_t + \rho(\mathbf{v}.\nabla \mathbf{v})\mathbf{v} = -\nabla p^\star + \mu_0(\mathbf{M}(|\mathbf{H_1}|).\nabla)\mathbf{H_1}$$

- Hence

$$\mathbf{v} = \nabla \phi, \qquad \Delta \phi = 0,$$

$$\rho \phi_t + \frac{\rho}{2}|\nabla \phi|^2 - \mu_0 \underbrace{\int_0^{|\nabla \psi_1|} s(\mu(s) - 1)\, ds}_{:= \nu(|\nabla \psi_1|)} + p^\star = c$$

- Dynamic condition at the free surface:
  - Balance of forces:



$$\tfrac{1}{2}\mu_0(\mathbf{M.n})^2 = \tfrac{1}{2}\mu_0(\mu(|\nabla \psi_1) - 1)^2 \psi_{1n}^2$$

  - $\psi = -J\theta/2\pi$, $\phi = 0$, $\zeta = 0$ is a solution

- Kinematic condition at the free surface:

$$(\partial_t + \mathbf{v}.\nabla)(r - R - \zeta) = 0 \quad \Rightarrow \quad -\zeta_t + \phi_r - \frac{1}{r^2}\phi_r \zeta_r - \phi_z \zeta_z = 0$$

AXISYMMETRIC WAVES

- For axisymmetric flows $\zeta = \zeta(z,t)$, $\phi = \phi(r,z,t)$ we always have

$$\psi = -\frac{J\theta}{2\pi}$$

  and we are left with a purely hydrodynamic problem.

- The dimensionless problem for travelling waves $\eta = \eta(z-ct)$, $\phi = \phi(z-ct)$ is

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad r < 1+\zeta,$$

$$\zeta_z + \phi_r - \phi_z\zeta_z = 0, \qquad r = 1+\zeta,$$

$$-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - a\left(\nu\left(\frac{1}{1+\zeta}\right) - \nu(1)\right)$$

$$+\beta\left(\frac{(1+\zeta_z^2)^{-\frac{1}{2}}}{(1+\zeta)} - \frac{\zeta_{zz}}{(1+\zeta_z^2)^{-\frac{1}{2}}} - 1\right) = 0, \qquad r = 1+\zeta$$

- Parameters: $a = \dfrac{\mu_0 J^2}{4\pi^2 R^2 c^2}$, $\beta = \dfrac{\sigma}{Rc^2}$

VARIATIONAL PRINCIPLE

- The variational principle

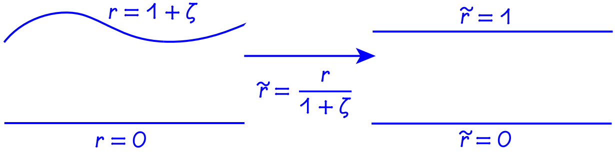
$$\delta \int \left\{ \int_0^{1+\zeta} \left(-r\phi_z + \frac{1}{2}(r\phi_r^2 + r\phi_z^2) \right) dr - aT(\zeta) + \beta(1+\zeta)(1+\zeta_z^2)^{\frac{1}{2}} - \tfrac{1}{2}\beta(1+\zeta)^2 \right\} dz = 0,$$

where

$$T(\zeta) = \int_0^{\zeta} \left(\nu\left(\tfrac{1}{1+s}\right) - \nu(1) \right) (1+s)\, ds,$$

recovers the hydrodynamic equations (with $r\phi_r|_{r=0} = 0$).

- Map the fluid domain to a cylinder:



- New variable: $\tilde{\phi}(\tilde{r}, z) = \phi(r, z)$
- New variational principle:

$$\delta \int L(\zeta, \tilde{\phi}, \zeta_z, \tilde{\phi}_z)\, dz = 0$$

SPATIAL DYNAMICS

- Variational principle: $\delta \int L(\zeta, \phi, \zeta_z, \phi_z)\, dz = 0$
- Legendre transform:

$$\omega = \frac{\delta L}{\delta \zeta_z},\ \psi = \frac{\delta L}{\delta \phi_z} \quad \Rightarrow \quad \zeta_z = \zeta_z(\zeta, \omega, \phi, \psi),\ \phi_z = \phi_z(\zeta, \omega, \phi, \psi)$$

- Hamiltonian:

$$\begin{aligned} H(\eta, \omega, \phi, \psi) &= \eta\zeta_z + \int_0^1 r\psi\phi_z - L(\eta, \phi, \eta_z, \phi_z), \\ &= \int_0^1 \left\{ \frac{1}{2}\left(\frac{\psi}{(1+\zeta)^2} + 1 \right)(1+\zeta)^2 r - \tfrac{1}{2} r\phi_r^2 \right\} dr \\ &\quad + aT(\zeta) - (1+\zeta)\sqrt{\beta^2 - W^2} + \tfrac{1}{2}\beta(1+\zeta)^2, \end{aligned}$$

where $W = \frac{1}{1+\zeta}\left(\omega + \frac{1}{1+\zeta} \int_0^1 r^2 \phi_r \psi \right)$

- Hamiltonian system:

$$\eta_z = \frac{\delta H}{\delta \omega}, \qquad \omega_z = -\frac{\delta H}{\delta \zeta}, \qquad \phi_z = \frac{\delta H}{\delta \psi}, \qquad \psi_z = -\frac{\delta H}{\delta \phi}$$

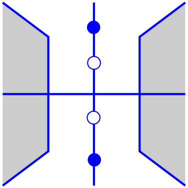
- Reversibility: $z \mapsto -z$, $(\eta, \omega, \phi, \psi) = (\eta, -\omega, -\phi, \psi)$

KIRCHGÄSSNER REDUCTION

Introduce bifurcation parameters $\alpha = \alpha_0 + \mu_1$, $\beta = \beta_0 + \mu_2$, so that

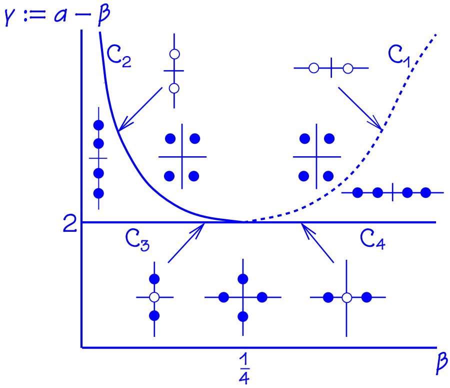
$$u_z = Lu + N_\mu(u) \qquad (\star)$$

Spectrum of L:



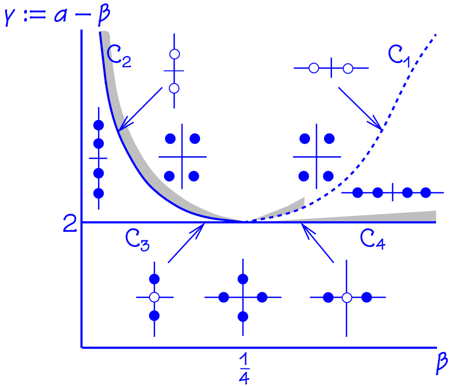
- $(\star)$ has a finite-dimensional invariant manifold M
- For small μ all small bounded solutions to $(\star)$ lie on M
- Reduced system:
  - describes the flow on M
  - is reversible and Hamiltonian
- Homoclinic solutions to the reduced system are solitary waves

EIGENVALUES



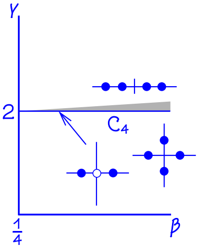
- An additional double zero eigenvalue due to the symmetry $\phi \mapsto \phi + c$

EIGENVALUES



- An additional double zero eigenvalue due to the symmetry $\phi \mapsto \phi + c$

KDV WAVES



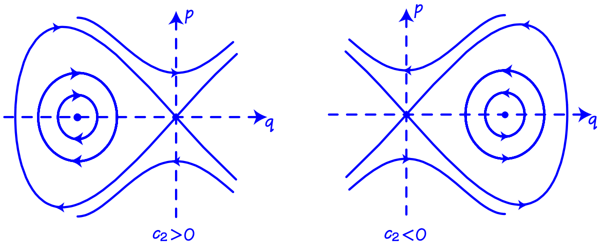
$$\beta = \beta_0 > \frac{1}{4},$$

$$\gamma = 2 + \delta^2$$

- Reduced system (after scaling):

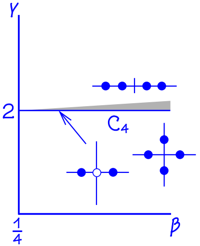
$$\begin{aligned} q_z &= p + O(\delta), \\ p_z &= q + c_2 q^2 + O(\delta) \end{aligned}$$

- Phase portrait:



- $c_2 > 0$ for $\beta < \beta_\star$, $c_2 < 0$ for $\beta > \beta_\star$

KDV WAVES



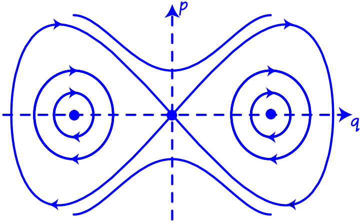
$$\beta = \beta_0 > \frac{1}{4},$$

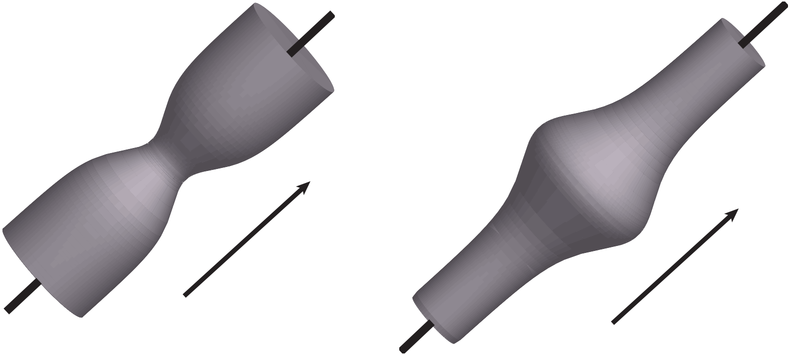
$$\gamma = 2 + \delta^2$$

- Reduced system (after scaling):

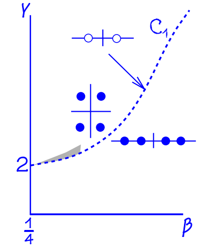
$$\begin{aligned} q_z &= p + O(\delta), \\ p_z &= q + \mu q^2 + c_3 q^3 + O(\delta) \end{aligned}$$

- Phase portrait:





KAWAHARA WAVES



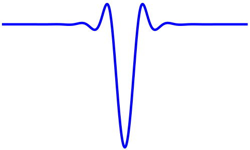
$$\beta = \tfrac{1}{4} + c(1+\mu)\delta^2,$$

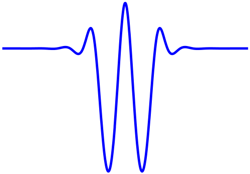
$$\gamma = 2 + c\delta^4$$

- Reduced system (after scaling):

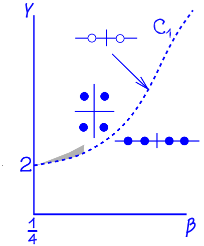
$$\begin{aligned} q_{1z} &= -p_1 + c_2 p_1^2 + O(\delta), \\ q_{2z} &= p_2 + O(\delta), \\ p_{1z} &= q_2 + O(\delta), \\ p_{2z} &= 2(1+\mu)q_2 + q_1 + O(\delta) \end{aligned}$$

- For δ, $\mu = 0$ we find $u'''' - 2u'' + u - u^2 = 0$ for $u \sim p_1$
- A family of multipulse solitary waves with exponentially decaying oscillatory tails (waves of elevation for $c_2 > 0$, waves of depression for $c_2 < 0$)





KAWAHARA WAVES



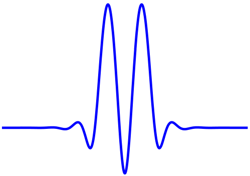
$$\beta = \tfrac{1}{4} + c(1+\mu)\delta^2,$$

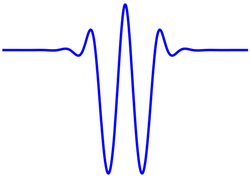
$$\gamma = 2 + c\delta^4$$

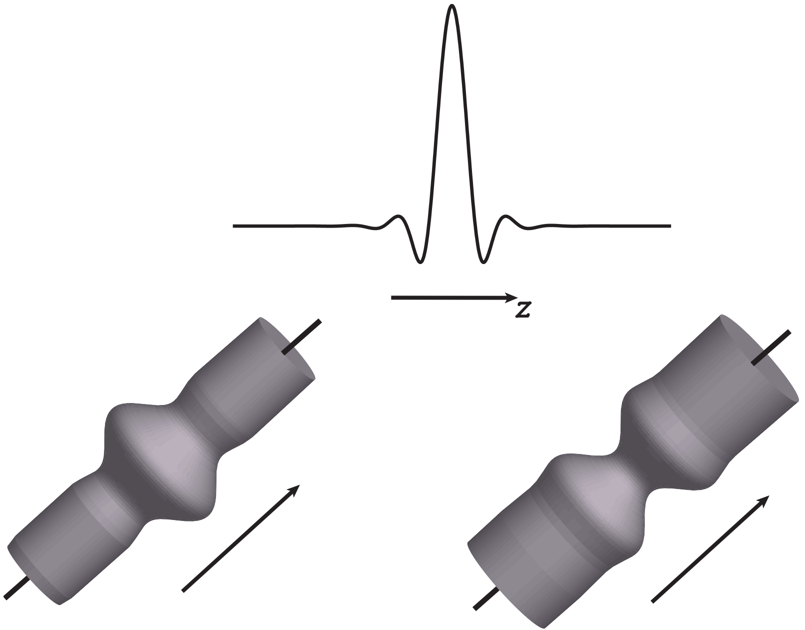
- Reduced system (after scaling):

$$\begin{aligned} q_{1z} &= -p_1 + \varepsilon p_1^2 + c_3 p_1^3 + O(\delta), \\ q_{2z} &= p_2 + O(\delta), \\ p_{1z} &= q_2 + O(\delta), \\ p_{2z} &= 2(1+\mu)q_2 + q_1 + O(\delta) \end{aligned}$$

- For δ, μ, $\varepsilon = 0$ we find $u'''' - 2u'' + u - u^3 = 0$ for $u \sim p_1$
- Two families of multipulse solitary waves with exponentially decaying oscillatory tails (waves of elevation, waves of depression)



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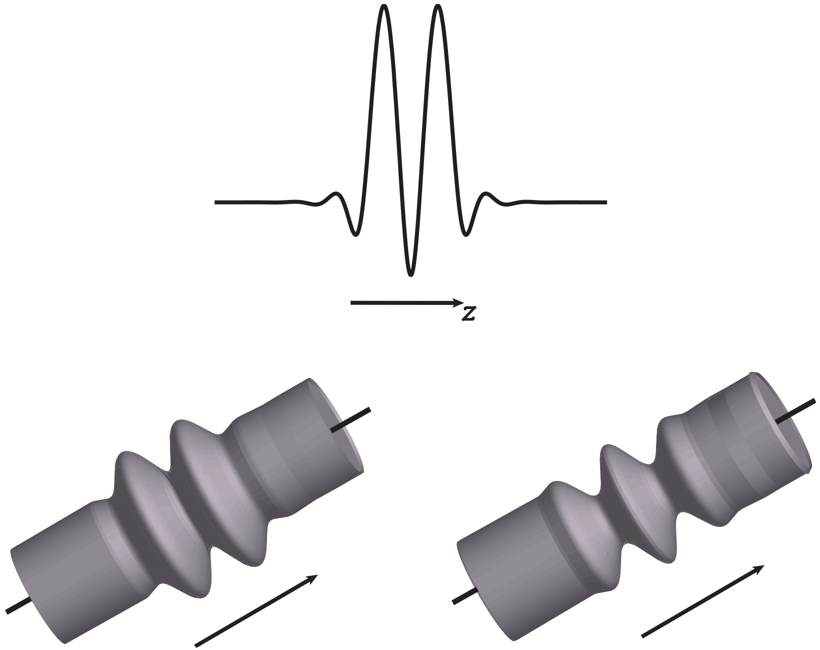
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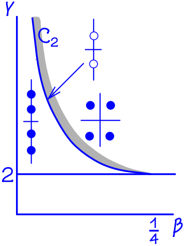
NLS WAVES



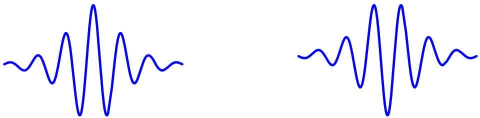
$$\beta = \beta_0,$$
$$\gamma = \gamma_0 + \delta^2$$
$$(\beta_0, \gamma_0) \in C_2$$

- Reduced system:

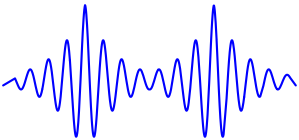
$$A_z = \frac{\partial H}{\partial \bar{B}}, \qquad B_z = -\frac{\partial H}{\partial \bar{A}},$$

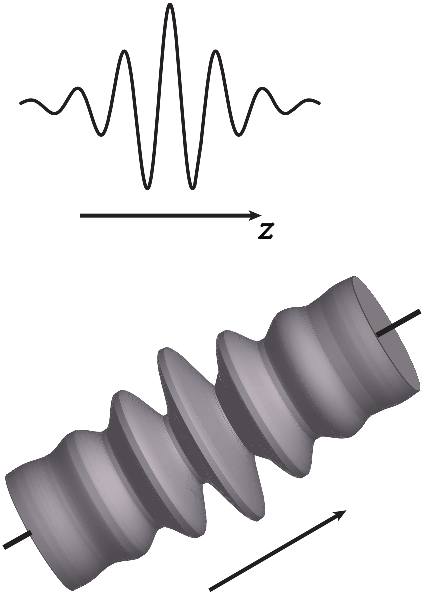
$$\begin{aligned} H = {} & i\omega(A\bar{B} - \bar{A}B) + |B|^2 \\ & + H_{NF}(|A|^2, i(A\bar{B} - \bar{A}B), \delta) \\ & + O(|(A,B)|^2 |(\delta, A, B)|^N) \end{aligned}$$

- Two symmetric envelope solitary waves:



- Multipulse envelope solitary waves:

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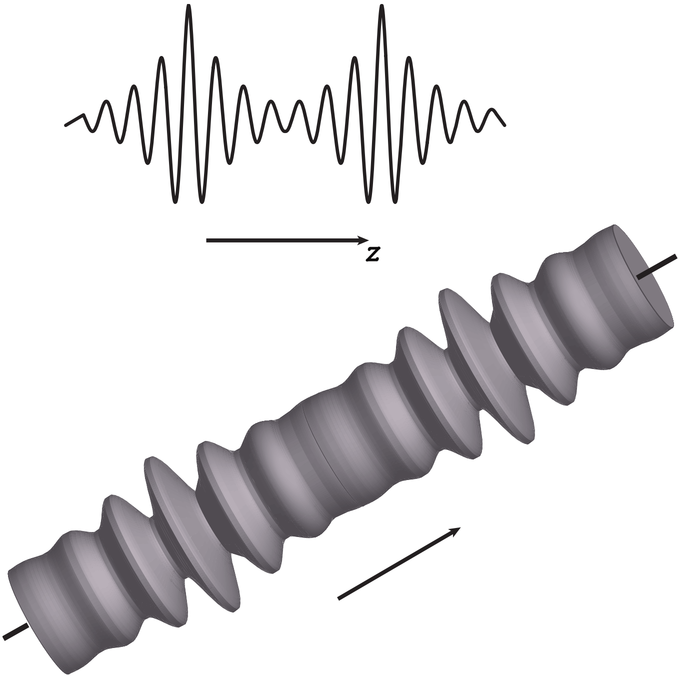
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