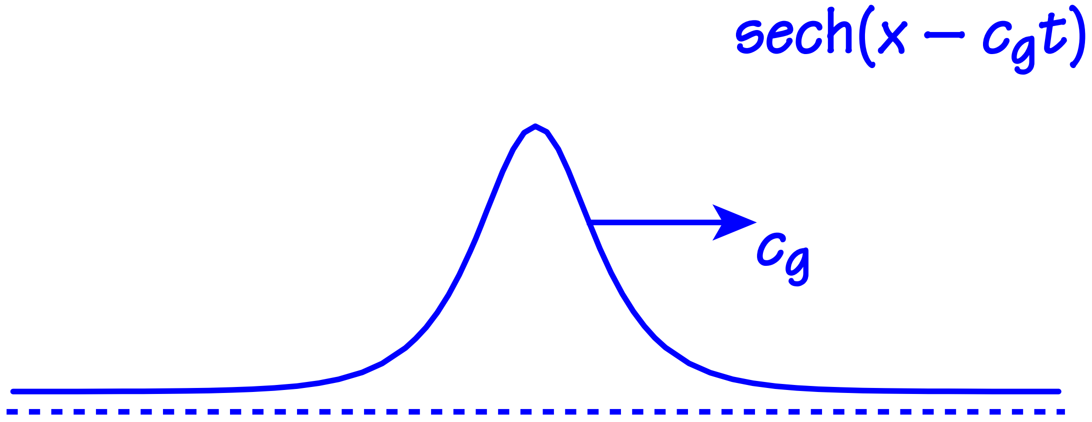
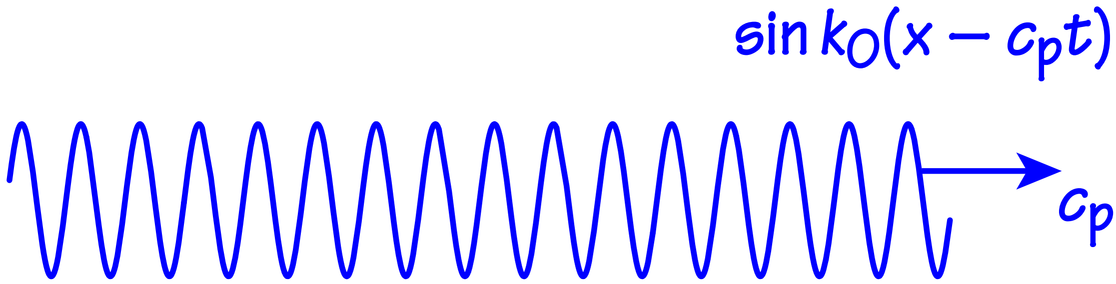
MODULATING PULSES

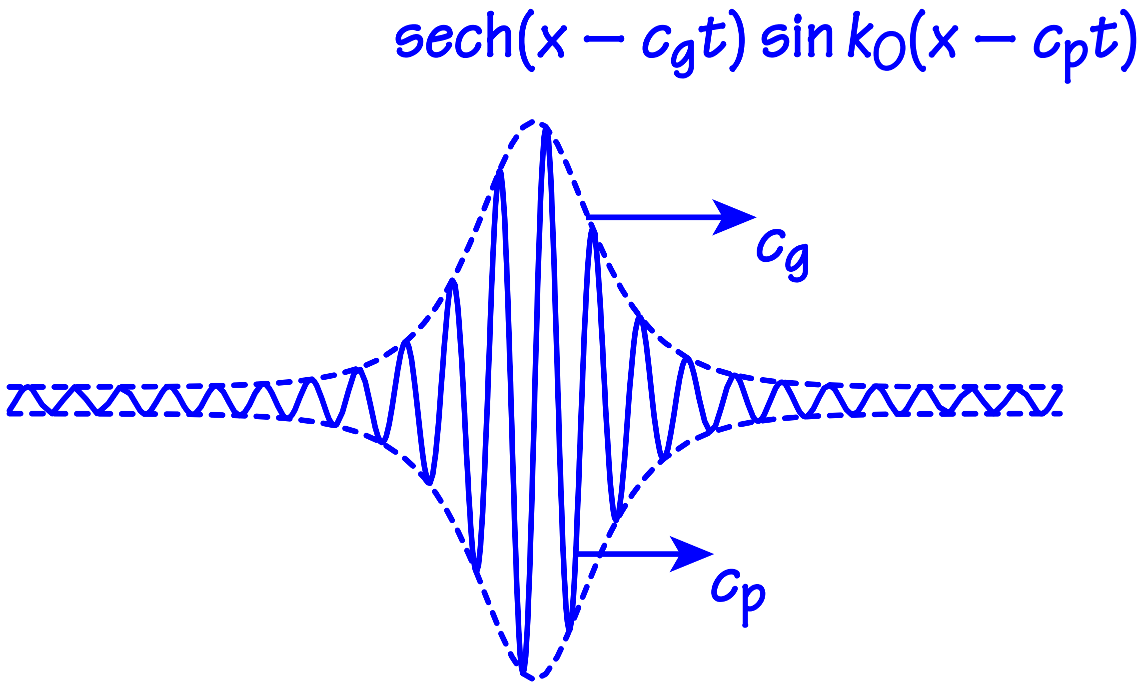
A pulse:



A wavetrain:



A modulating pulse:



WAVE EQUATIONS

We study the equation

$$\partial_t^2 u = \partial_x^2 u - u + f_1(u, \partial_x u, \partial_t u)\partial_x^2 u + f_2(u, \partial_x u, \partial_t u),$$

where

- f_1, f_2 are analytic
- $f_i(a, -b, -c) = f_i(a, b, c), \qquad i = 1, 2$

Linear dispersion relation for $\sin k(x - c_p' t)$:

$$c_p' = (1 + k^2)^{1/2}/k$$

$$c_g' = \frac{d}{dk}(kc_p') = \frac{1}{c_p'}$$

We look for modulating pulse solutions of the form

$$u(x, t) = v_1(x - c_g t, k_0(x - c_p t)),$$
$$\text{“}\mathrm{sech}(x - c_g t) \sin k_0(x - c_p t)\text{”}$$

where

- $v_1(\xi, \eta)$ is 2π-periodic in η
- $c_p = c_p' + \gamma_1 \varepsilon^2, \qquad 0 < \varepsilon \ll 1$
- $c_g = 1/c_p$

SPATIAL DYNAMICS

We formulate the equation as a system for

$$v = (v_1, v_2), \qquad v_2 = \partial_\xi v_1,$$

so that

$$\begin{aligned} \partial_\xi v_1 &= v_2, \\ \partial_\xi v_2 &= -c_3^\varepsilon k_0^2 \partial_\eta^2 v_1 - c_4^\varepsilon v_1 \\ &\qquad + g_0^\varepsilon(v) \partial_\eta^2 v_1 + g_1^\varepsilon(v) + g_2^\varepsilon(v) \partial_\eta v_2, \end{aligned}$$

where

$$g_j^\varepsilon(v) = g_j^\varepsilon(v_1, \partial_\eta v_1, v_2), \qquad j = 1, 2, 3.$$

Phase space:

- $X^s = H^{s+1}_{\text{per}}(0, 2\pi) \times H^s_{\text{per}}(0, 2\pi), \qquad s > 0$
- The vector field has domain X^{s+1}
- $g_0^\varepsilon, g_1^\varepsilon, g_2^\varepsilon$ are bounded $X^s \longrightarrow H^s_{\text{per}}(0, 2\pi)$

Reversibility:

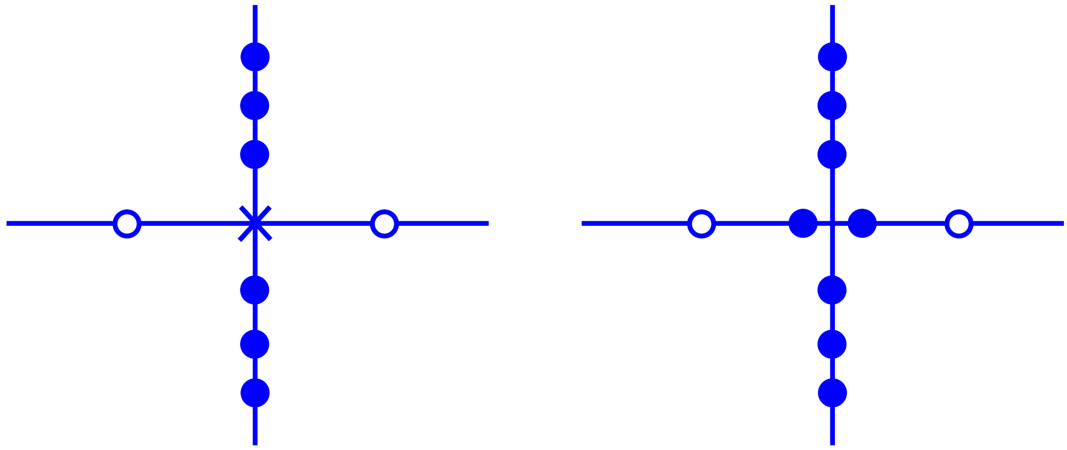
- The system is invariant under

$$\xi \mapsto -\xi, \qquad (v_1, v_2) \mapsto S(v_1, v_2),$$

$$S(v_1(\eta), v_2(\eta)) = (v_1(-\eta), -v_2(-\eta))$$

- Symmetric solutions are invariant under this transformation

SPECTRAL ANALYSIS



$\varepsilon = 0$ $\qquad\qquad$ $\varepsilon > 0$

- Infinitely many imaginary eigenvalues $\pm i\omega_m$, $\omega_m \sim m$

Notation:

$$z = P_{wh}(v), \qquad q = P_{sh,c}(v)$$

Write as a coupled system

$$\partial_\xi z = Kz + F^\varepsilon(z,q), \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\partial_\xi q_1 = q_2,$$

$$\partial_\xi q_2 = -c_3^\varepsilon k_0^2 \partial_\eta^2 q_1 - c_4^\varepsilon q_1 + \overbrace{P_{sh,c}(g_1^\varepsilon(z,q))}^{= g_3^\varepsilon(z,q) + h^\varepsilon(z)}$$
$$+ P_{sh,c}(g_0^\varepsilon(z,q)\partial_\eta^2 q_1) + P_{sh,c}(g_2^\varepsilon(z,q)\partial_\eta q_2),$$

where $g_3^\varepsilon(z,0) = 0$.

A SIMPLIFIED PROBLEM

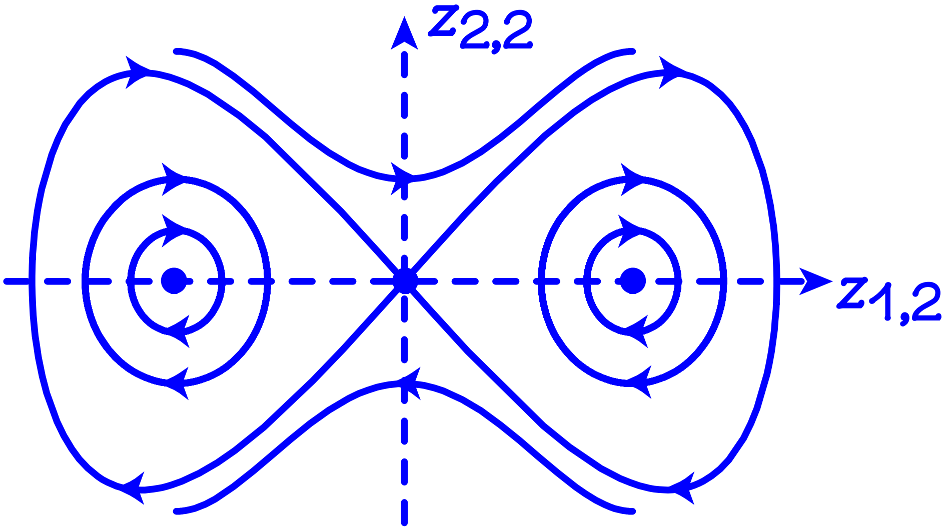
- $h^\varepsilon(z) = 0 \;\Rightarrow\; \{q = 0\}$ is invariant
- The flow in $\{q=0\}$ is controlled by the fourth-order equation

$$\partial_\xi z = Kz + \underbrace{F^\varepsilon(z,0)}_{\text{cubic}},$$

Use scaled variables:

$$\begin{aligned}
\partial_\xi z_{1,1} &= z_{2,1},\\
\partial_\xi z_{2,1} &= C_1 z_{1,1} - C_2 z_{1,1}(z_{1,1}^2 + z_{1,2}^2) + O(\varepsilon),\\
\partial_\xi z_{1,2} &= z_{2,2},\\
\partial_\xi z_{2,2} &= C_1 z_{1,2} - C_2 z_{1,2}(z_{1,1}^2 + z_{1,2}^2) + O(\varepsilon)
\end{aligned}$$

For $\varepsilon = 0$ we have a two-dimensional invariant plane containing two symmetric homoclinics:



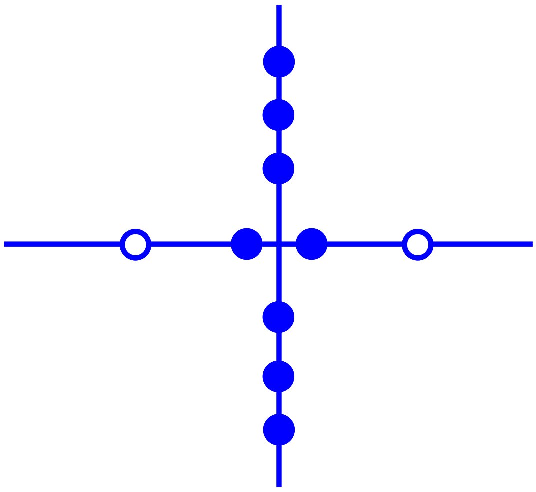
Two symmetric homoclinics p^ε persist for $\varepsilon > 0$ and are estimated by

$$|p^\varepsilon(\xi)| \le c\varepsilon e^{-\varepsilon|\xi|}$$

A NONEXISTENCE RESULT

The homoclinics found in $\{q = 0\}$ generically do not persist for $h^{\varepsilon}(z) \neq 0$

- Dynamical systems arguments:



Three-dimensional stable and unstable manifolds

The manifolds generically do not intersect in an infinite-dimensional phase space

- Global existence theory for one-dimensional quadratic, quasilinear wave equations?

Use a change of variable to make $h^{\varepsilon}(z)$ 'small'

Prepare: The scaling $\varepsilon = \mu^2$ and

$$(z_1', z_2', q_1', q_2') = (\mu^{-1} z_1, \mu^{-3} z_2, \mu^{-2} q_1, \mu^{-2} q_2)$$

yields

$$\partial_\xi z = Kz + F^{\varepsilon}(z, q), \qquad |p^{\varepsilon}(\xi)| \leq c\varepsilon e^{-\varepsilon|\xi|}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\partial_\xi z' = F^{\mu}(z', q'), \qquad |p^{\mu}(\xi)| \leq c\mu e^{-\mu^2|\xi|}$$

NORMAL-FORM THEORY

$$\partial_\xi z = F^\mu(z,q),$$

$$\partial_\xi q_1 = q_2 + g_6^\mu(z,q) + h_1^\mu(z),$$
$$\partial_\xi q_2 = -c_3^\mu k_0^2 \partial_\eta^2 q_1 - c_4^\mu q_1 + g_4^\mu(z,q) + h_2^\mu(z)$$
$$+ P_{sh,c}(g_3^\mu(z,q)\partial_\eta^2 q_1) + P_{sh,c}(g_3^\mu(z,q)\partial_\eta q_2)$$

Try to eliminate the term $h^n(z,\mu)$ by substituting

$$\tilde{q} = q + \Phi^n(z,\mu)$$

Homological equation:

$$L^n \Phi^n(z,\mu) = h^n(z,\mu),$$

$$L^n \begin{pmatrix} \Phi^n_{1,k} \\ \Phi^n_{2,k} \end{pmatrix} z^i \mu^j = \begin{pmatrix} 0 & 1 \\ \omega_k^2 & 0 \end{pmatrix} \begin{pmatrix} \Phi^n_{1,k} \\ \Phi^n_{2,k} \end{pmatrix} z^i \mu^j, \quad i + j = n$$

This equation can always be solved:

- $\Phi^n = (L^n)^{-1} h^n$
- $\|(L^n)^{-1}\| \leq a$
- Gain of one derivative

The change of variable preserves the structure, and there is an optimal choice of n such that

$$\|h^\mu(z)\| \leq c\mu^2 e^{-c^\star/\mu}$$

CENTRE-STABLE MANIFOLD

We look for solutions of the form

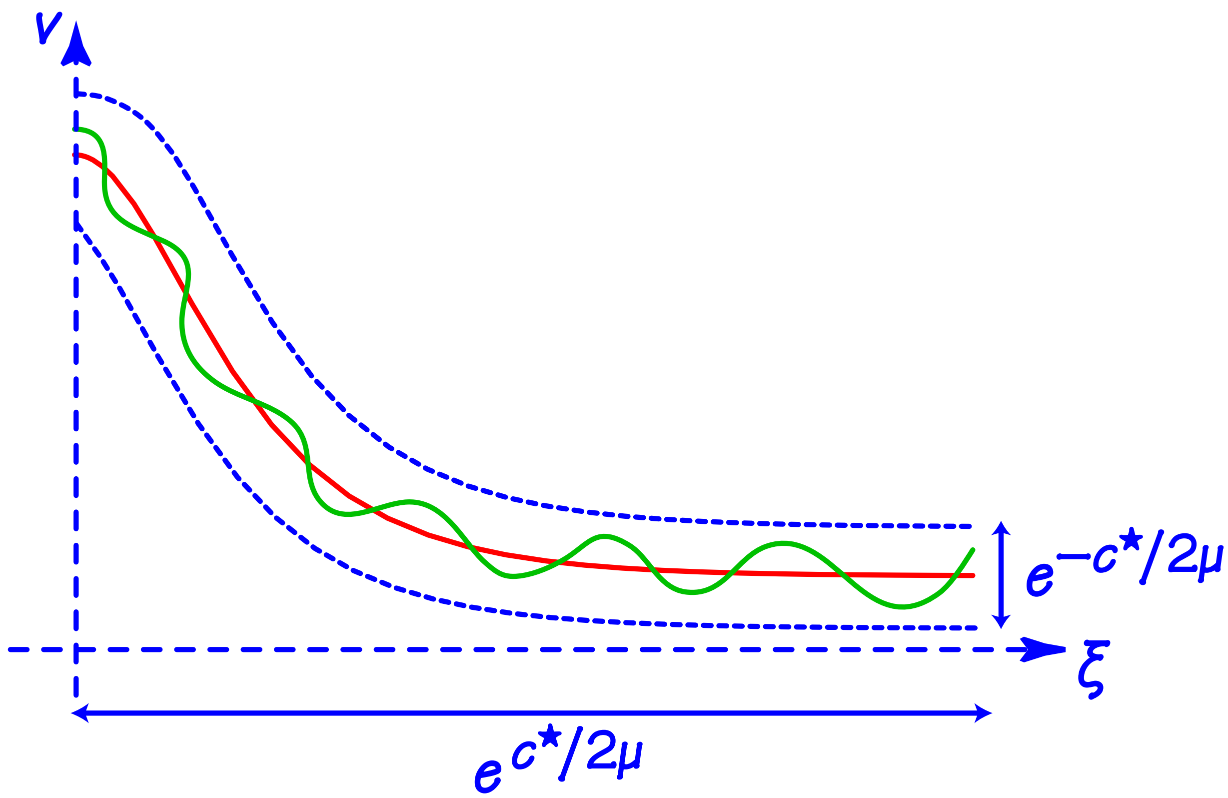
$$(z, q) = (p^{\mu} + Z, q), \quad (Z, q) \in C([0, e^{c^\star/2\mu}], X^{s+1})$$

with

$$\|(Z(\xi), q(\xi))\| \leq e^{-c^\star/2\mu}$$

Combine

- Dynamical-systems arguments
- Kato's iteration scheme



"Centre-stable manifold":

- Define $W^{cs} = \{v(0) \text{ for such solutions } v\}$
- W^{cs} is given as a graph $v_u(0) = f(v_s(0), v_c(0))$

EXISTENCE THEORY

Consider

$$\begin{aligned}\partial_\xi w_1 &= w_2,\\ \partial_\xi w_2 &= -c_3^\mu k_0^2 \partial_\eta^2 w_1 - c_4^\mu w_1\\ &\qquad + \mu P_c((w_1 + \mu e^{-\mu^2\xi})\partial_\eta w_2) + h^\mu\end{aligned}$$

in the phase space $P_c X^s$.

We show that this system has a unique solution $w \in C([0, e^{c^\star/2\mu}], P_c X^{s+1})$, with

$$\|w(\xi)\| \le e^{-c^\star/2\mu}, \qquad \xi \in [0, e^{c^\star/2\mu}].$$

- Use the Kato iteration scheme

$$\begin{aligned}\partial_\xi w_{2(m+1)} &= w_{2(m+1)},\\ \partial_\xi w_{2(m+1)} &= -c_3^\mu k_0^2 \partial_\eta^2 w_{1(m+1)} - c_4^\mu w_{1(m+1)}\\ &\qquad + \mu P_c((w_{1(m)} + \mu e^{-\mu^2\xi})\partial_\eta w_{2(m+1)}) + h^\mu\end{aligned}$$

  with $(w_{1(0)}, w_{2(0)}) = (0,0)$ and initial data

$$(w_{1(m+1)}(0), w_{2(m+1)}(0)) = (w_1^0, w_2^0)$$

- Show that

$$\|w_{(m)}(\xi)\| \le e^{-c^\star/2\mu} \;\Rightarrow\; \|w_{(m+1)}(\xi)\| \le e^{-c^\star/2\mu},$$

$$\|w_{(m+1)}(\xi) - w_{(m)}(\xi)\| \le \frac{1}{2}\|w_{(m)}(\xi) - w_{(m-1)}(\xi)\|$$

  for $\xi \in [0, e^{c^\star/2\mu}]$.

EXISTENCE THEORY

$$\partial_\xi w_{2(m+1)} = w_{2(m+1)},$$
$$\partial_\xi w_{2(m+1)} = -c_3^\mu k_0^2 \partial_\eta^2 w_{1(m+1)} - c_4^\mu w_{1(m+1)}$$
$$+ \mu P_c((w_{1(m)} + \mu^2 e^{-\mu^2 \xi})\partial_\eta w_{2(m+1)}) + h^\mu$$

Define

$$E(w) = \int \{(\partial_\eta^{s+1} w_2)^2 - c_3^\mu k_0^2 (\partial_\eta^{s+2} w_1)^2 + c_4^\mu (\partial_\eta^{s+1} w_1)^2\}\, d\eta$$

- $E(w)^{1/2}$ is the norm on $P_c X_{s+1}$
- Show that

$$\|w_{(m)}(\xi)\| \leq e^{-c^\star/2\mu} \Rightarrow \|w_{(m+1)}(\xi)\| \leq e^{-c^\star/2\mu}$$

Apply $\partial_\eta^{s+1} w_{2(m+1)} \partial_\eta^{s+1}$ to the second equation and integrate with respect to η:

$$\partial_\xi E(w_{(m+1)}) \leq c\mu e^{-c^\star/\mu} E(w_{(m+1)})^{1/2}$$
$$+ c\mu(e^{-c^\star/2\mu} + \mu^2 e^{-\mu^2\xi}) E(w_{(m+1)})$$

Integration yields

$$\sup_{\xi \in [0, e^{c^\star/2\mu}]} E(w_{(m+1)}(\xi))$$
$$\leq E(w_{(m+1)}(0)) + c\mu e^{-c^\star/\mu} + c\mu \sup_{\xi \in [0, e^{c^\star/2\mu}]} E(w_{(m+1)}(\xi))$$

and hence

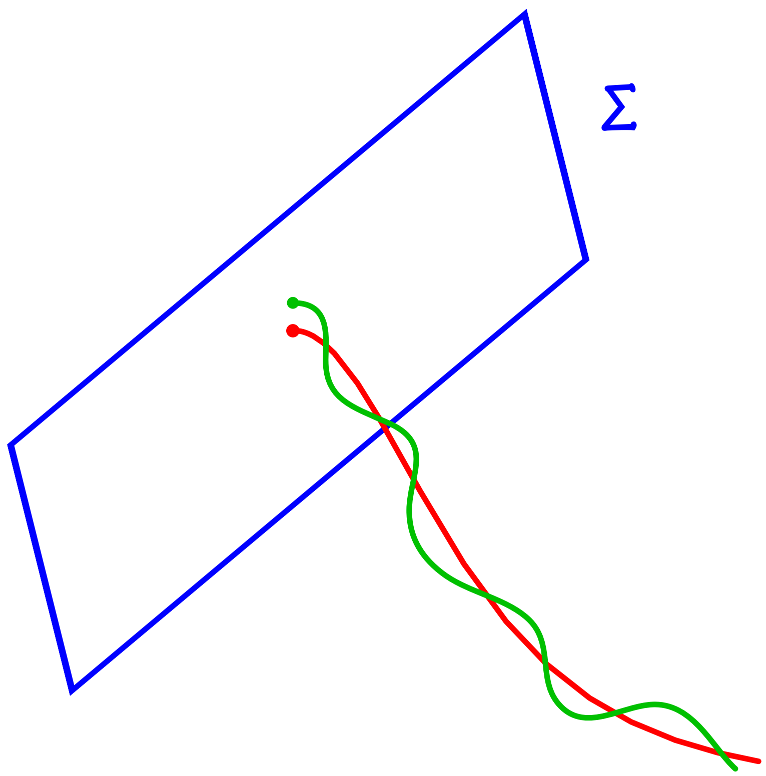
$$\sup_{\xi \in [0, e^{c^\star/2\mu}]} E(w_{(m+1)}(\xi)) \leq e^{-c^\star/\mu}.$$

SYMMETRIC PULSES

- Define the "symmetric section" $\Sigma = \operatorname{Fix} S$
- A solution $v(\xi)$ on $[0, e^{c^\star/2\mu}]$ with $v(0) \in \Sigma$ can be extended to a symmetric solution

$$\tilde{v}(\xi) = \begin{cases} v(\xi), & \xi \geq 0 \\ Sv(-\xi), & \xi < 0 \end{cases}$$

on $[-e^{c^\star/2\mu}, e^{c^\star/2\mu}]$.



- Solve

$$(I - S)v(0) = 0, \qquad v(0) \in W^{cs}$$

perturbatively around $p^\mu(0)$.

- W^{cs} intersects Σ in an infinite family of points (parameterised by $P_c(I - S)v(0)$)