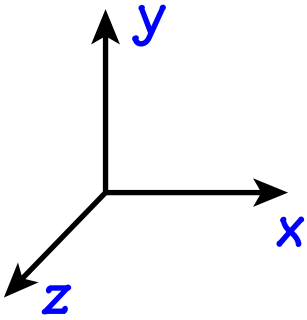
THE WATER-WAVE PROBLEM

$$y = h + \eta(x, z, t)$$



Velocity potential ϕ

Kinematic boundary condition:

$$\eta_t = \phi_y - \eta_x \phi_x - \eta_z \phi_z$$

Dynamic boundary condition:

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \sigma \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x - \sigma \left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z = 0$$

Difficulties:

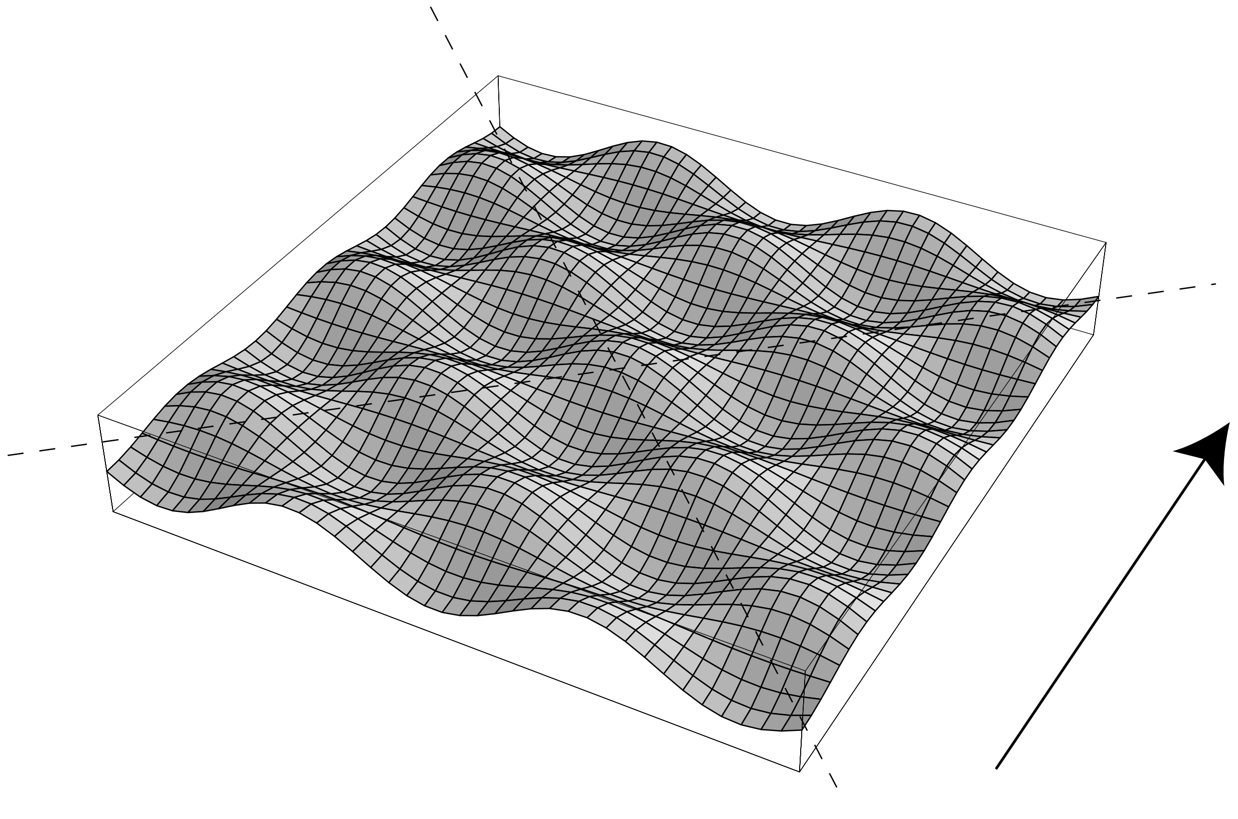
- A free boundary problem
- Nonlinear boundary conditions

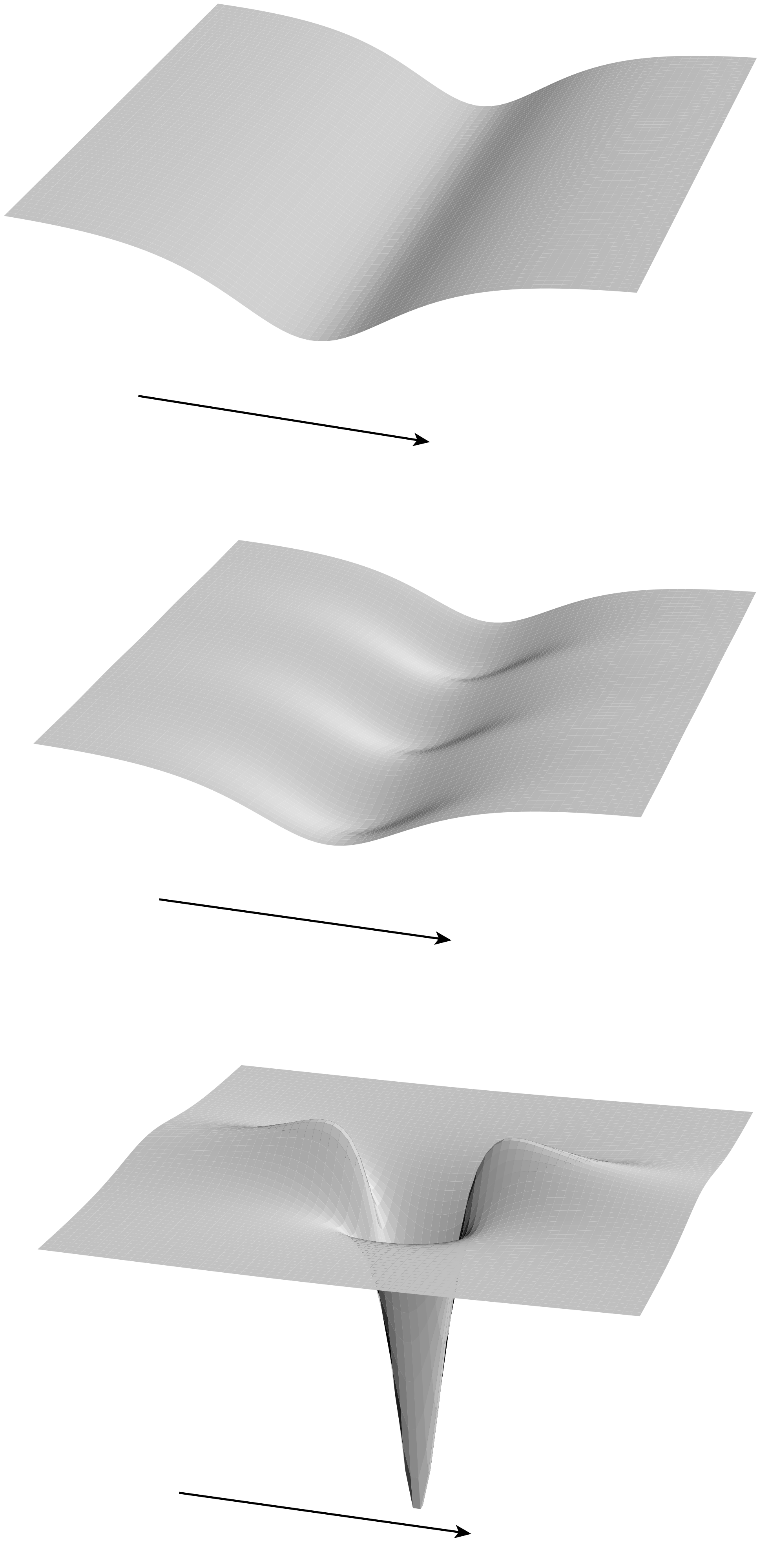
Travelling waves:

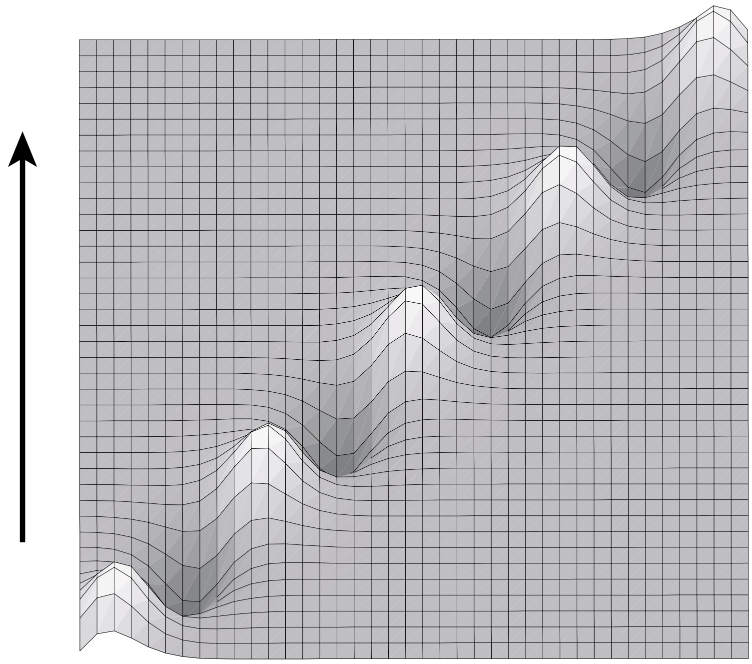
$$\eta(x, z, t) = \eta(x - ct, z), \ \phi(x, y, z, t) = \phi(x - ct, y, z)$$

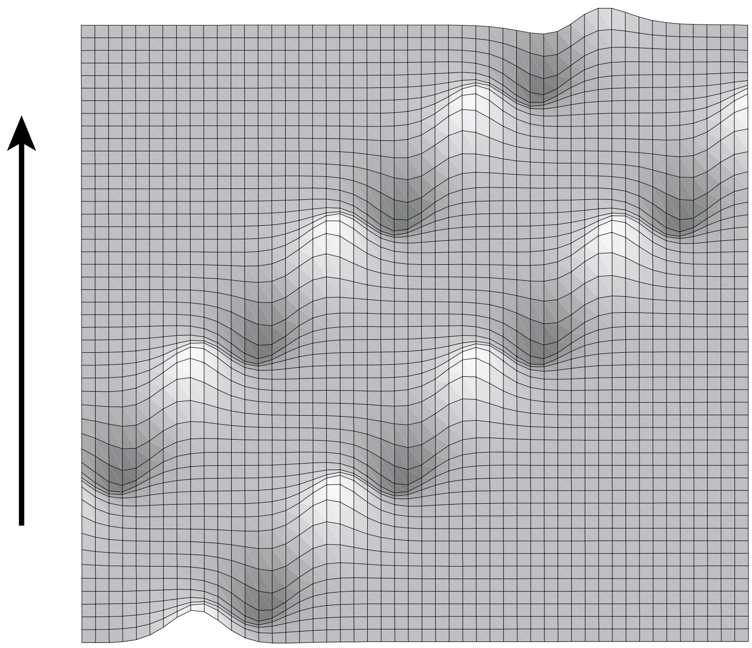
Parameters:

$$\alpha = gh/c^2, \quad \beta = \sigma/hc^2$$









VARIATIONAL PRINCIPLE

Luke's variational principle:

$$\delta \int_{\mathbb{R}^2} \left\{ \int_0^{1+\eta} \left(-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right) dy + \frac{1}{2} a \eta^2 + \beta \left(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1 \right) \right\} dx\, dz = 0$$

New variables:

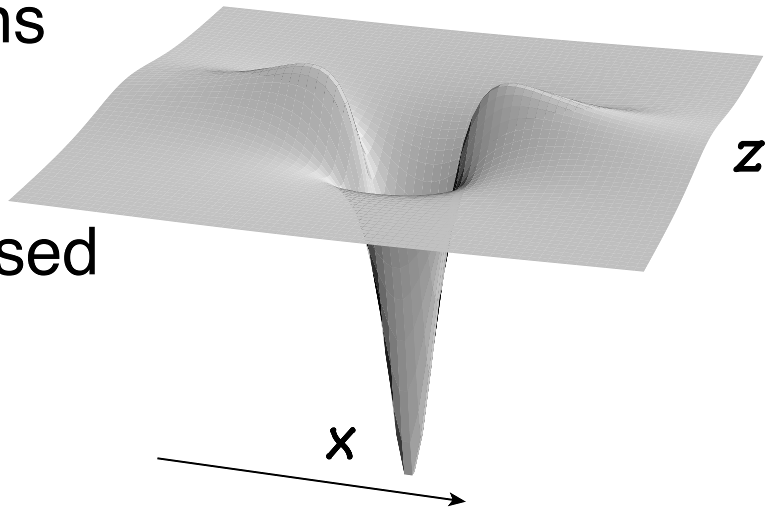
$$\tilde{y} = y/(1 + \eta(x,z)),$$
$$\phi(x,y,z) = \Phi(x,\tilde{y},z)$$

New variational principle:

$$\delta \int_{\mathbb{R}^2} \left\{ \int_0^1 \left(\frac{1}{2} \left(\Phi_x - \frac{y \eta_x \Phi_y}{1+\eta} \right)^2 + \frac{\Phi_y^2}{2(1+\eta)^2} + \frac{1}{2} \left(\Phi_z - \frac{y \eta_z \Phi_y}{1+\eta} \right)^2 \right) (1+\eta)\, dy + \eta \Phi_x|_{y=1} + \frac{1}{2} a \eta^2 + \beta \left[\sqrt{1 + \eta_x^2 + \eta_z^2} - 1 \right] \right\} dx\, dz = 0$$

- Apply the direct methods of the calculus of variations

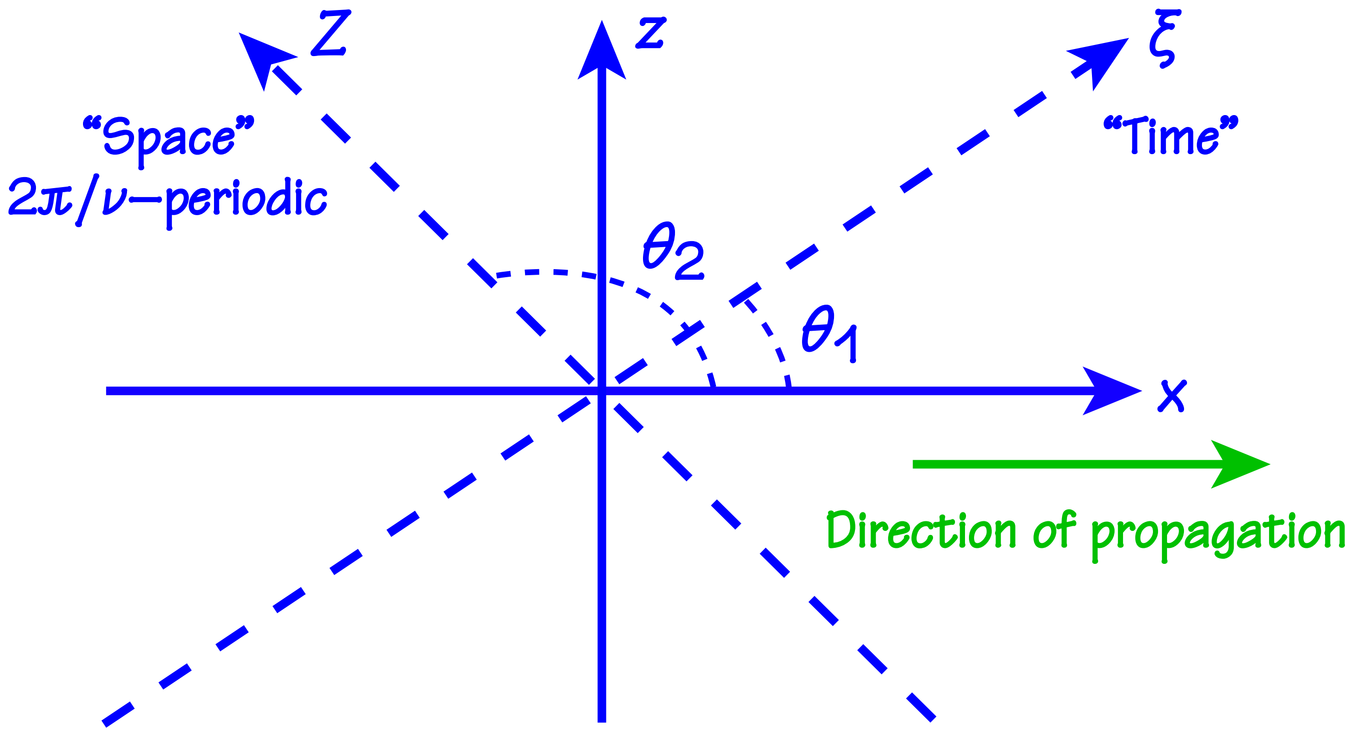
Yields fully localised solitary waves



SPATIAL DYNAMICS

Variational principle:

$$\delta\int_{\mathbb{R}^2}\left\{\int_0^1\left(\frac{1}{2}\left(\Phi_x-\frac{y\eta_x\Phi_y}{1+\eta}\right)^2+\frac{\Phi_y^2}{2(1+\eta)^2}+\frac{1}{2}\left(\Phi_z-\frac{y\eta_z\Phi_y}{1+\eta}\right)^2\right)(1+\eta)\,dy\right.$$
$$\left.+\eta\Phi_x|_{y=1}+\frac{1}{2}\alpha\eta^2+\beta[\sqrt{1+\eta_x^2+\eta_z^2}-1]\right\}dx\,dz=0$$



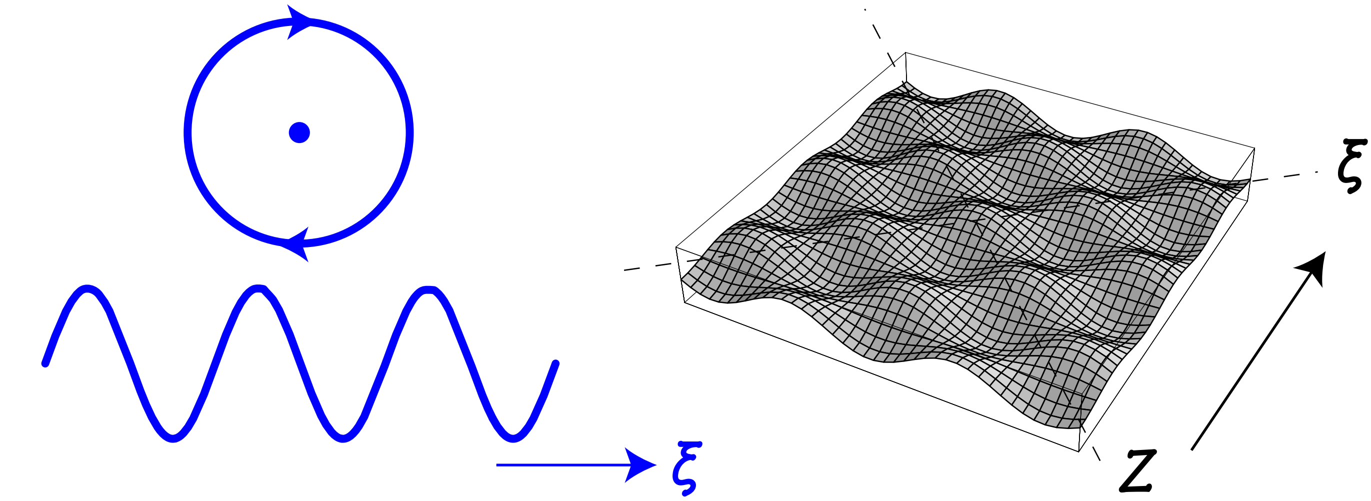
Perform the Legendre transform:

$$\delta\int f(\eta,\Phi,\eta_\xi,\Phi_\xi)\,d\xi=0$$

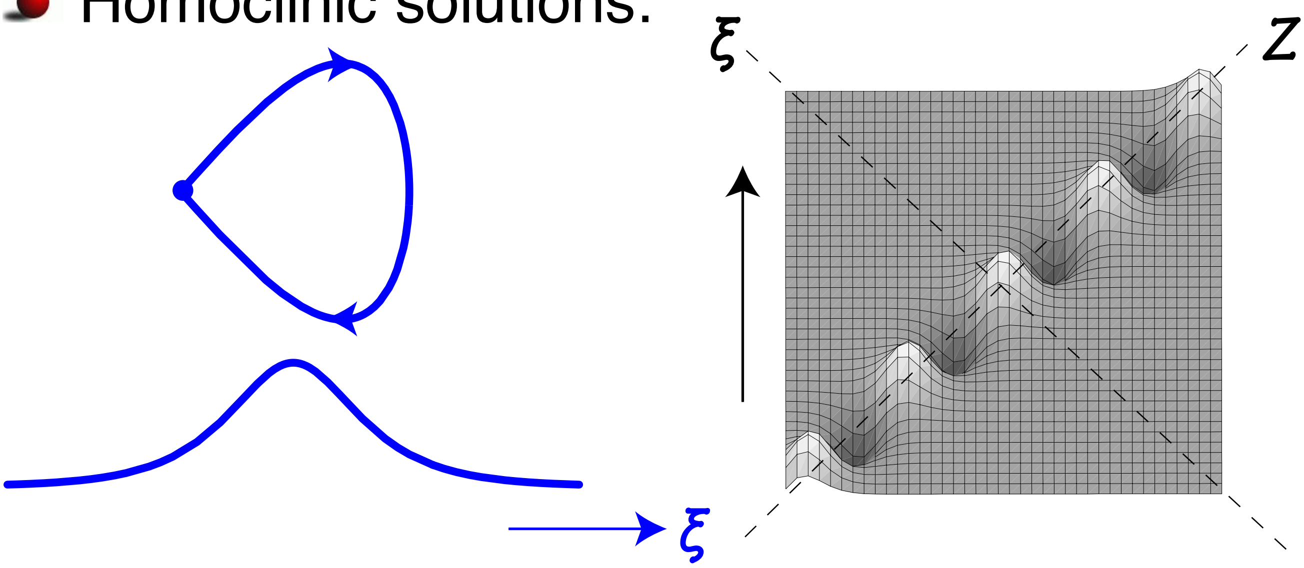
$$u_\xi=Lu+N(u),\qquad u=(\eta,\Phi,\partial_{\eta_\xi}f,\partial_{\Phi_\xi}f)$$

SOLUTIONS OF THE EVOLUTIONARY EQUATION

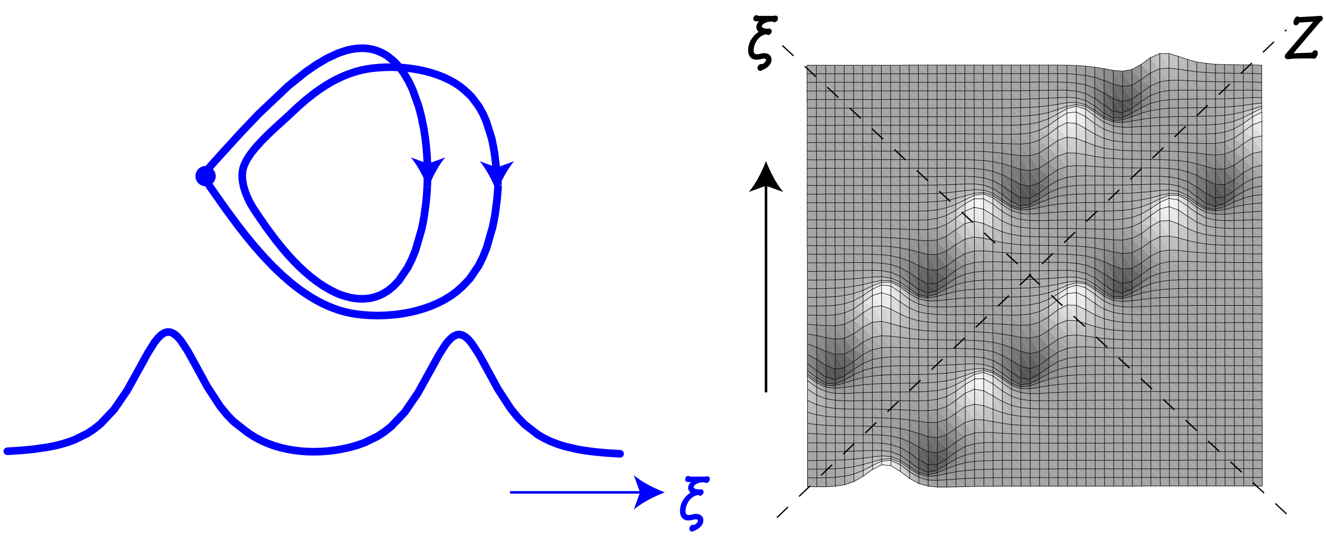
- Periodic solutions:



- Homoclinic solutions:



- Multi-pulse homoclinic solutions:



THE CENTRE MANIFOLD

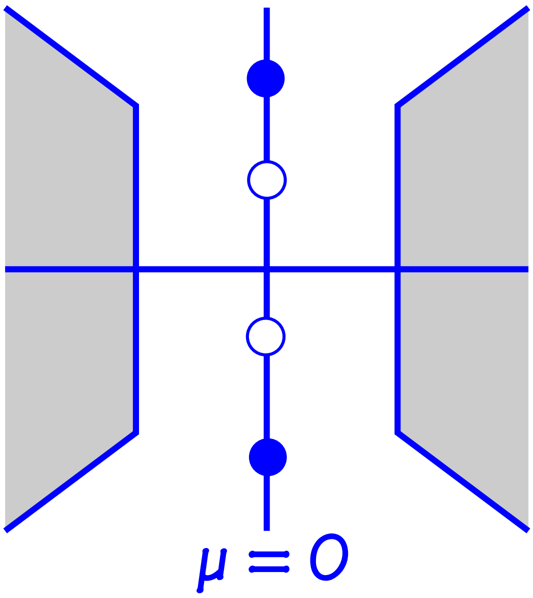
Introduce bifurcation parameters

$$(a, \beta, \nu, \theta_1, \theta_2) = (a_0, \beta_0, \nu_0, \theta_{10}, \theta_{20}) + \mu,$$

so that

$$u_\xi = L_\mu u + N_\mu(u) \qquad (\star)$$

Spectrum:



- $(\star)$ has a finite-dimensional invariant manifold M
- For small μ all small, bounded solutions to $(\star)$ lie on M
- Reduced system:
  - describes the flow on M
  - has a Hamiltonian structure

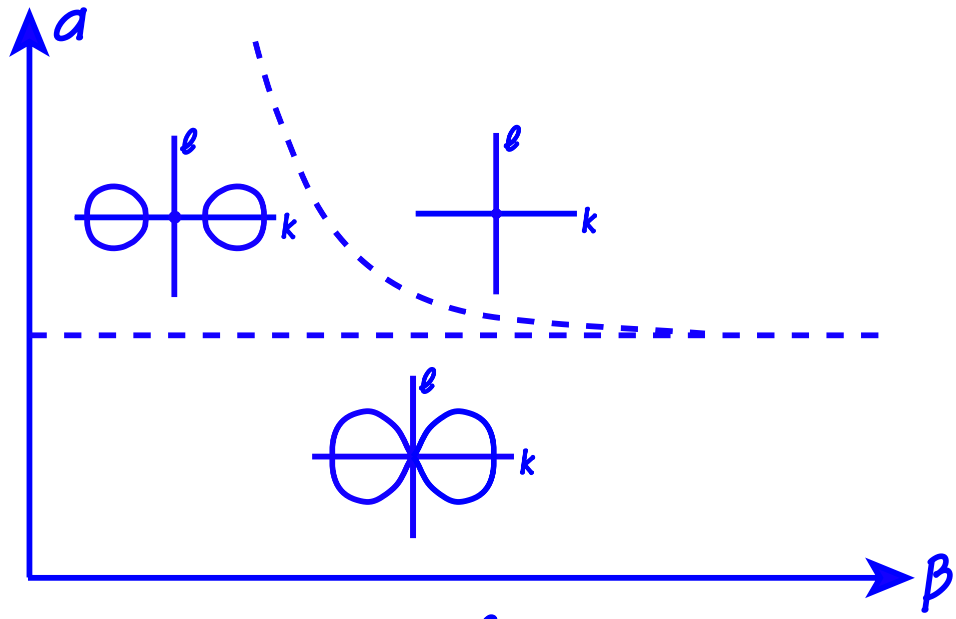
IMAGINARY EIGENVALUES

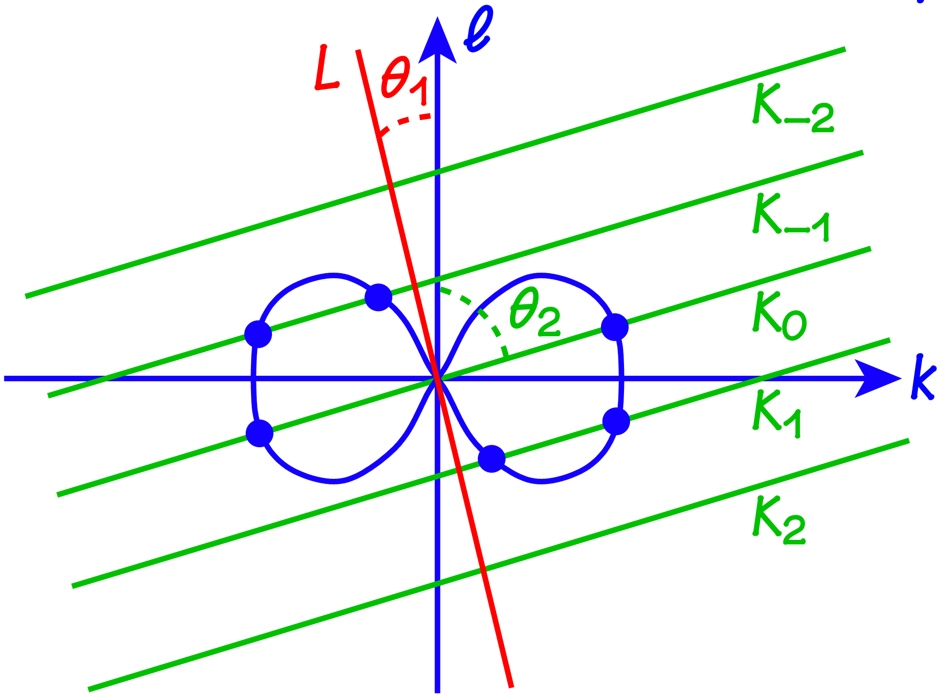
A 'mode n' eigenvalue $i\kappa$ arises when

$$\eta(\xi, Z) \sim e^{i\kappa\xi} e^{in\nu Z}$$
$$= e^{ikx} e^{i\ell z}, \qquad \left.\begin{aligned} k &= \kappa \sin\theta_2 + n\nu \sin\theta_1 \\ \ell &= -\kappa \cos\theta_2 - n\nu \cos\theta_1 \end{aligned}\right\} K_n$$

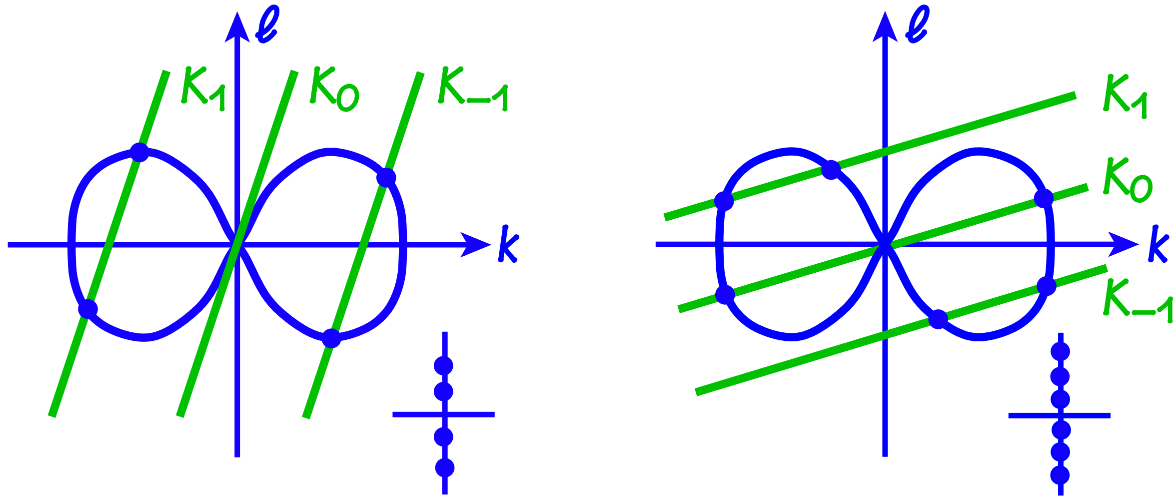
und exists when k, ℓ satisfy the dispersion relation

$$-k^2 + (a + \beta(k^2 + \ell^2))\sqrt{k^2 + \ell^2} \tanh \sqrt{k^2 + \ell^2} = 0$$

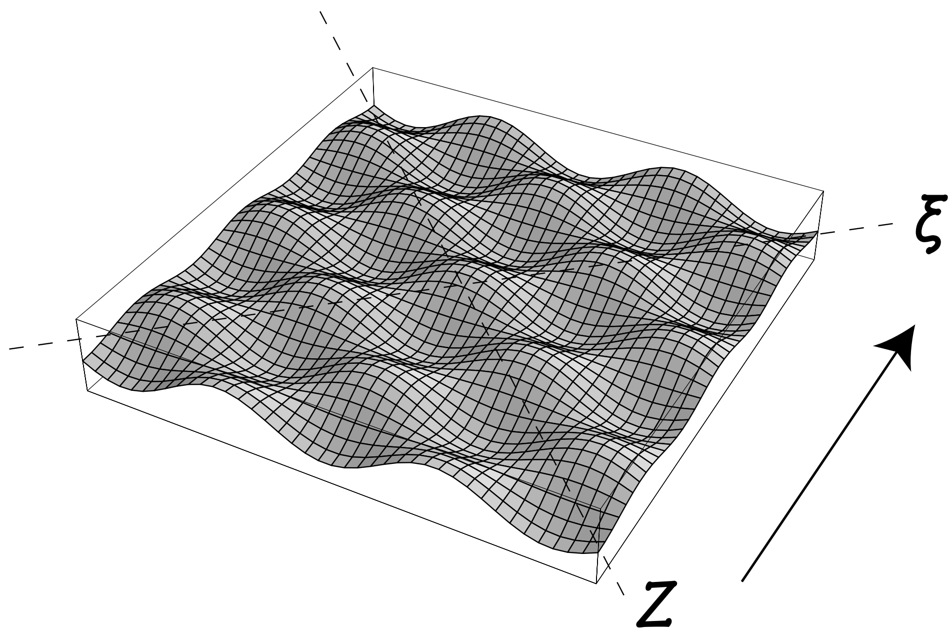




PERIODIC WAVES

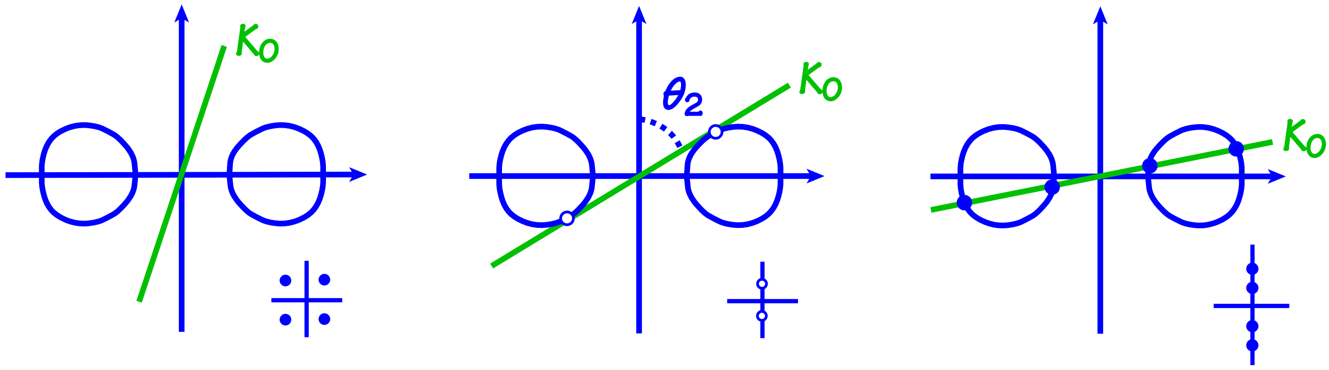


- A reduced Hamiltonian system with finitely many degrees of freedom
- Periodic solutions are found using the Lyapunov centre theorem (many resonant cases)
- Inverse result (Groves & Haragus, Craig & Nicholls): Given any angles θ_1, θ_2 and frequencies κ, ν there is a corresponding doubly periodic wave

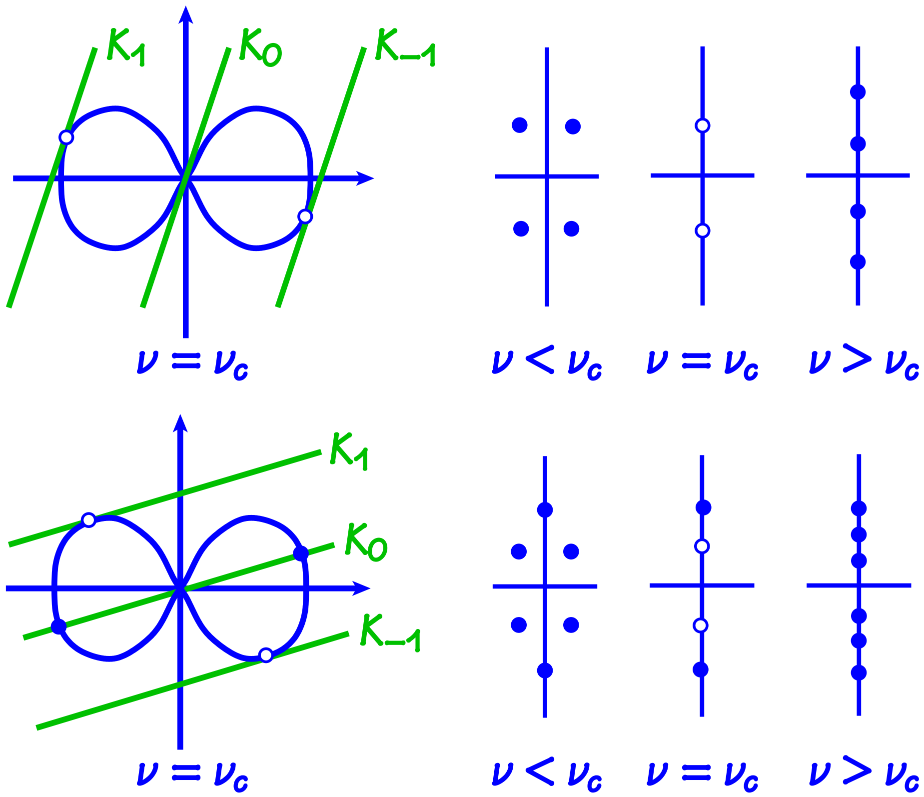


BIFURCATIONS

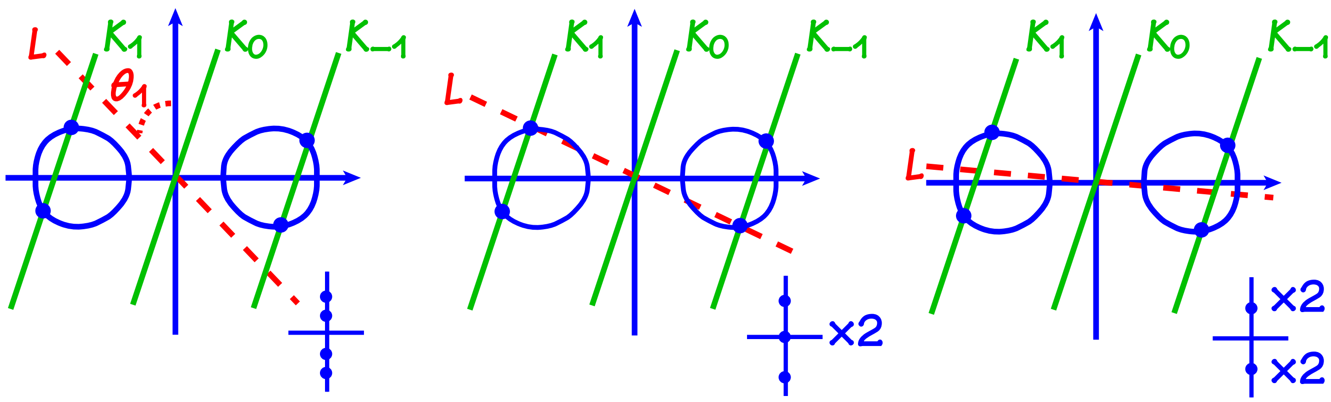
- Vary θ_2:



- Vary ν:

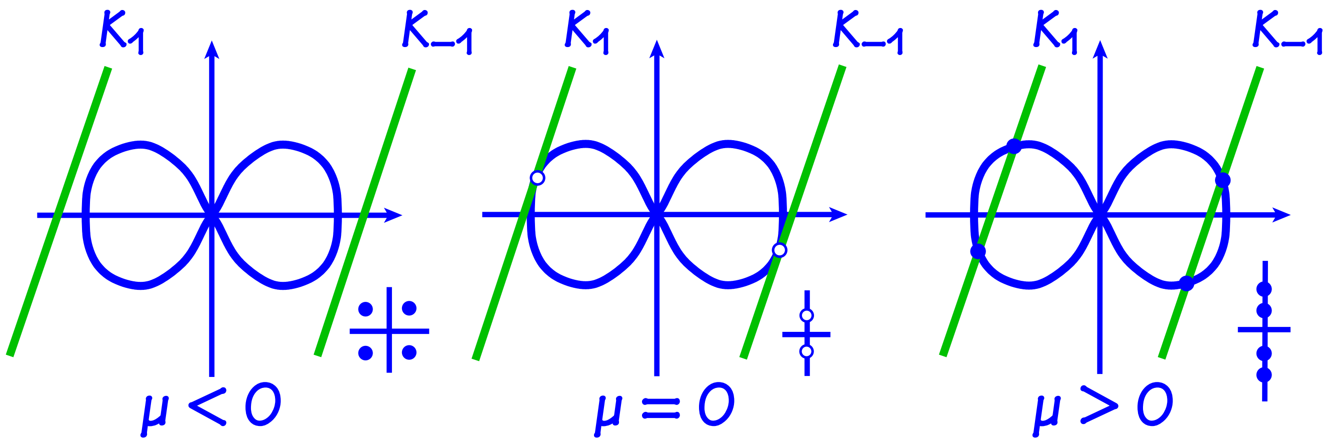


- Vary θ_1:



- Practically every bifurcation and eigenvalue resonance possible (Groves & Haragus)

HAMILTONIAN HOPF BIFURCATION



Iooss & Kirchgässner:

- Introduce complex coordinates and use a partial normal-form transformation:

$$\dot{A} = \frac{\partial H}{\partial \bar{B}}, \qquad \dot{B} = -\frac{\partial H}{\partial \bar{A}}$$

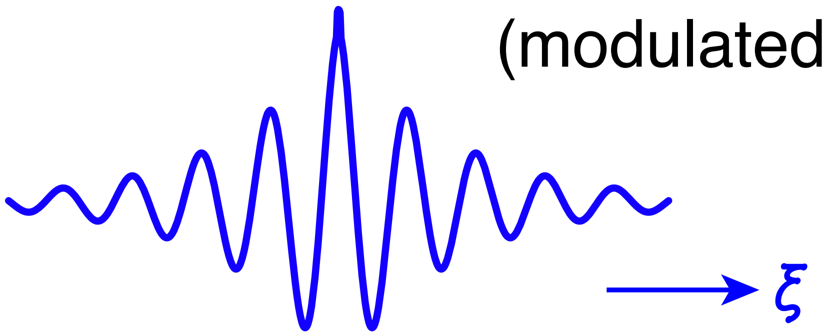
$$H = i\kappa(A\bar{B} - \bar{A}B) + |B|^2 + \underbrace{\tilde{H}(\mu, |A|^2, i(A\bar{B} - \bar{A}B))}_{\substack{\text{Normal form} \\ \text{Polynomial of order } N+1}} + \underbrace{O(N+2)}_{\substack{\text{Remainder} \\ \text{term}}}$$

- Without the remainder terms:
  - Complete integrability
  - A circle of homoclinic solutions for $\mu > 0$
- Two homoclinic solutions are symmetric and survive re-introduction of the remainder terms

HOMOCLINIC SOLUTIONS

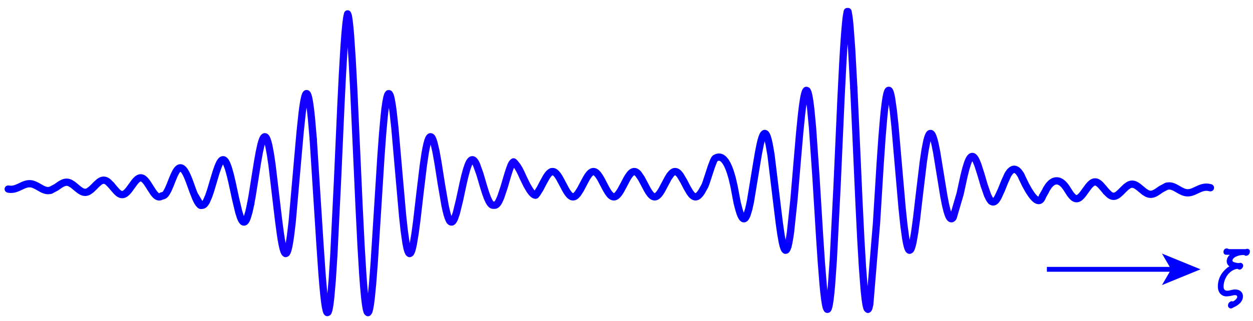
Iooss & Kirchgässner: A pair of symmetric homoclinic solutions

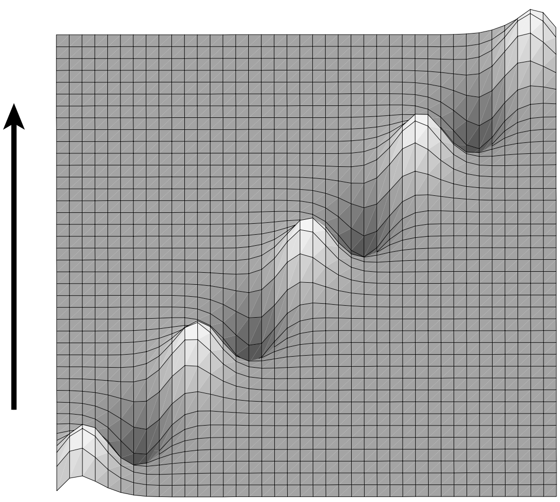
(modulated waves)

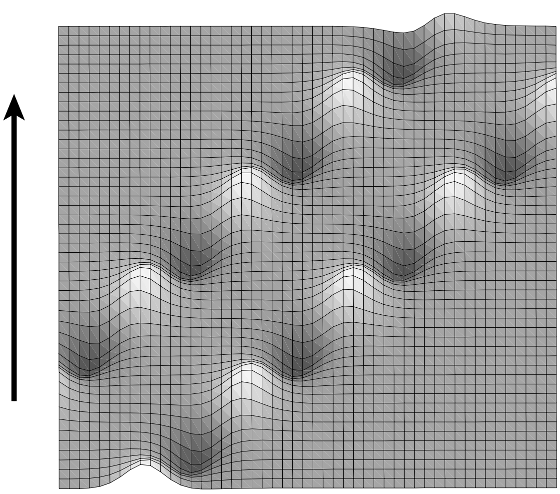


Buffoni & Groves:

- Homoclinic solutions are critical points of a functional of ‘mountain-pass’ type
- Use the calculus of variations and toplogical degree to construct multipulse solutions







FULLY LOCALISED SOLITARY WAVES

- Apply the direct methods of the calculus of variations to the original variational principle:

$$\delta\int_{\mathbb{R}^2}\left\{\int_0^1\left(\frac{1}{2}\left(\Phi_x-\frac{y\eta_x\Phi_y}{1+\eta}\right)^2+\frac{\Phi_y^2}{2(1+\eta)^2}+\frac{1}{2}\left(\Phi_z-\frac{y\eta_z\Phi_y}{1+\eta}\right)^2\right)(1+\eta)\,dy\right.$$
$$\left.+\eta\Phi_x|_{y=1}+\frac{1}{2}a\eta^2+\beta[\sqrt{1+\eta_x^2+\eta_z^2}-1]\right\}dx\,dz=0$$

- Quasilinear structure
- Groves & Sun: Reduce to a locally equivalent seimilinear problem using a generalisation of the variational Lyapunov-Schmidt reduction procedure