This is a seminar on the existence of rational curves on algebraic varieties. It is a seminal result of Mori that if a smooth projective variety $X$ has a curve $C$ such that $K_X \cdot C < 0$, then it has a rational curve $\tilde{C}$ such that $K_X \cdot \tilde{C} < 0$. This is the beginning of a story which eventually led to his Fields medal in 1990. The existence of rational curves on a variety makes the geometry of a variety richer and more interesting, and many classes of varieties have (many) rational curves on them, such as uniruled and rationally connected varieties, and in particular, Fano manifolds.

In this Seminar we follow [Deb01] in order to construct such rational curves. The story is a beautiful blend of different techniques, such as deformation of varieties and morphisms, reduction to positive characteristic and, occasionally, cohomological methods. We will start with deformation techniques, calculating dimensions of certain relevant deformation spaces, and then see how reduction modulo $p$ allows to make these dimensions large enough in order to produce rational curves. We will proceed to see how Fano manifolds have many rational curves, and in particular, each two points on a (complex) Fano manifold can be joined by a rational curve, quite analogously to the case of projective spaces. If time permits, we will prove Mori’s Cone theorem on the structure of the cone of effective curves on a manifold.

The seminar takes place during the Winter term 2015/16 on Tuesdays 12–14, in Seminar room 1.007. The preliminary meeting took place on 14 July. If you have any additional questions, please send an email to lazic@math.uni-bonn.de. Two weeks before each talk, the corresponding participant should briefly discuss their topic with me.

**Prerequisites:** We assume basic concepts of algebraic geometry, e.g. as explained in [Har77, Chapter II]. Some knowledge of cohomology as at the beginning of [Har77, Chapter III] is useful. The program follows [Deb01] closely; other possible sources are [Kol96, KM98], and also [Laz04] for a beautiful introduction to positivity of divisors on varieties. All the references in the following list of talks are as in [Deb01].

0. **Organisational talk** (27.10.15)

1. **Q- and $\mathbb{R}$-divisors, 1-cycles, intersection numbers** (03.11.15, Lazić)
   Assigned reading: Sections 1.1, 1.2, Subsection 1.11
   Talk: Define divisors with rational and real coefficients and 1-cycles. Define the intersection product of Cartier divisors via the Euler characteristic and show
that it is a multilinear symmetric function. Prove the projection formula. Define the cone of curves.

2. **Nakai-Moishezon criterion for ampleness and nef divisors** (10.11.15, Luca Tasin)
   
   Assigned reading: Subsection 1.20, Sections 1.5, 1.6
   

3. **Kleiman's criterion for ampleness and asymptotic Riemann-Roch** (17.11.15, Enrica Floris)
   
   Assigned reading: Sections 1.7, 1.8, 1.9
   
   Talk: Prove Kleiman’s criterion for ampleness and Proposition 1.31. Present some examples from Section 1.9.

4. **Exceptional locus of a morphism and rational curves** (24.11.15, Domenico Valloni)
   
   Assigned reading: Sections 1.10, 1.11
   
   Talk: Define the exceptional locus and discuss its properties. Prove Propositions 1.43 and 1.45.

5. **Parametrising rational curves and morphisms, I** (01.12.15, Salvatore Flocari)
   
   Assigned reading: Sections 2.1, 2.2 (until p. 43)
   
   Talk: Introduce sets \( \text{Mor}(X, Y) \) and discuss their properties. Discuss in detail the special case \( X = \mathbb{P}^1 \). Prove Proposition 2.4 and Theorem 2.6, assuming Lemmas 2.7 and 2.8.

6. **Parametrising rational curves and morphisms, II** (08.12.15, Paul Görlach)
   
   Assigned reading: Sections 2.2 (from p. 43), 2.3, 2.4 (until p. 49)
   
   Talk: Prove Lemmas 2.7 and 2.8. Discuss morphisms fixing a subscheme. Prove Proposition 2.13.

7. **Producing rational curves and reduction modulo \( p \)** (15.12.15, Domenico Valloni)
   
   Assigned reading: Section 3.1, pp. 62–63
   
   Talk: Show Propositions 3.1 and 3.2 (bend and break). Describe what it means
to reduce a variety to positive characteristic and give properties of the reductions.

8. **Rational curves on Fano manifolds** (12.01.16, Thi Quynh Trang Nguyen)

Assigned reading: Sections 3.2, 3.3

Talk: Prove that a Fano manifold possesses a rational curve through every point. Prove a stronger version of bend and break, Proposition 3.5.

9. **Rational curves on varieties** $X$ **such that** $K_X$ **is not nef** (19.01.16, Shanxiao Huang)

Assigned reading: Sections 3.4, 4.1

Talk: Prove that a projective variety $X$ such that $K_X$ is not nef possesses a rational curve. Discuss uniruled and rationally connected varieties.

10. **The Cone theorem** (26.01.16, Paul Görlach)

Assigned reading: Sections 6.1, 6.3, 6.4

Talk: State and prove Mori's cone theorem.

11. **Consequences of the Cone theorem and the Minimal Model Program** (02.02.16, Lazić)

Assigned reading: Sections 6.2, 6.5

Talk: Discuss some consequences of the Cone theorem as in Section 6.2. Discuss the Minimal Model Program, and prove Proposition 6.10.

**References**


