The Configuration Space of Amoebas with barycentric Simplex Newton Polytope

Timo de Wolff
joint with T. Theobald

GOETHE UNIVERSITÄT
FRANKFURT AM MAIN

5th of July 2011

Introduction

Notation

Notation

- We investigate Laurent polynomials in $\mathbb{C}\left[\mathbf{z}^{\pm 1}\right]$ with $\mathbf{z} := z_1 \cdots z_n$.

Notation

- We investigate Laurent polynomials in $\mathbb{C}\left[\mathbf{z}^{\pm 1}\right]$ with $\mathbf{z} := z_1 \cdots z_n$.
- A Laurent polynomial has the form $f := \sum_{j=1}^{d} b_j \cdot \mathbf{z}^{\alpha(j)}$.

Notation

- We investigate Laurent polynomials in $\mathbb{C}\left[\mathbf{z}^{\pm 1}\right]$ with $\mathbf{z} := z_1 \cdots z_n$.
- A Laurent polynomial has the form $f := \sum_{j=1}^{d} b_j \cdot \mathbf{z}^{\alpha(j)}$.
- The VARIETY of a Laurent polynomial f is denoted as $\mathcal{V}(f)$.

Notation

- We investigate Laurent polynomials in $\mathbb{C}\left[\mathbf{z}^{\pm 1}\right]$ with $\mathbf{z} := z_1 \cdots z_n$.
- A Laurent polynomial has the form $f := \sum_{j=1}^{d} b_j \cdot \mathbf{z}^{\alpha(j)}$.
- The VARIETY of a Laurent polynomial f is denoted as $\mathcal{V}(f)$.
- For fixed exponents we denote the corresponding set as

$$A := \{\alpha(j) \in \mathbb{Z}^n : 1 \leq j \leq d\}.$$

Notation

- We investigate Laurent polynomials in $\mathbb{C}\left[\mathbf{z}^{\pm 1}\right]$ with $\mathbf{z} := z_1 \cdots z_n$.
- A Laurent polynomial has the form $f := \sum_{j=1}^{d} b_j \cdot \mathbf{z}^{\alpha(j)}$.
- The VARIETY of a Laurent polynomial f is denoted as $\mathcal{V}(f)$.
- For fixed exponents we denote the corresponding set as

$$A := \{\alpha(j) \in \mathbb{Z}^n : 1 \leq j \leq d\}.$$

- The NEWTON POLYTOPE $\mathrm{New}(f)$ of a Laurent polynomial f is the polytope obtained by taking the convex hull of all its exponent vectors, i.e. here we have $\mathrm{New}(f) = \mathrm{conv}\, A$.

What is an amoeba?

What is an amoeba?

Definition (Gelfand, Kapranov, Zelevinsky)

Let $f \in \mathbb{C}[\mathbf{z}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$.

What is an amoeba?

Definition (Gelfand, Kapranov, Zelevinsky)

Let $f \in \mathbb{C}[\mathbf{z}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. Define the Log-map as:

$$\begin{aligned} \mathrm{Log}: \quad & (\mathbb{C}^*)^n \to \mathbb{R}^n, \\ & (|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|) \end{aligned}$$

What is an amoeba?

Definition (Gelfand, Kapranov, Zelevinsky)

Let $f \in \mathbb{C}[\mathbf{z}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. Define the Log-map as:

$$\begin{aligned} \mathrm{Log}: \quad & (\mathbb{C}^*)^n \to \mathbb{R}^n, \\ & (|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|) \end{aligned}$$

The AMOEBA $\mathcal{A}(f)$ of f is the image of $\mathcal{V}(f)$ under the Log-map.

What is an amoeba?

Example

What is an amoeba?

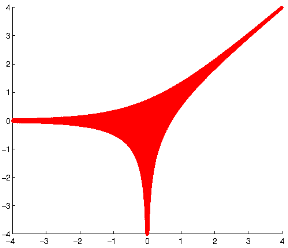
Example

1. $f := z_1 + z_2 + 1$

What is an amoeba?

Example

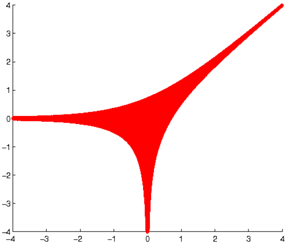
1. $f := z_1 + z_2 + 1$



What is an amoeba?

Example

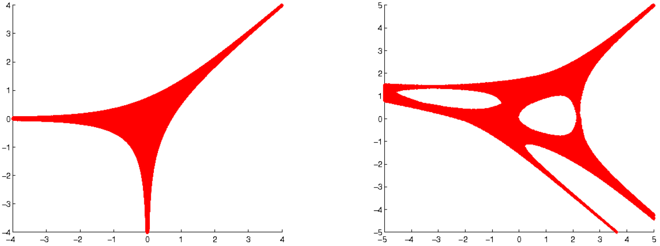
1. $f := z_1 + z_2 + 1$
2. $f := z_1^3 z_2^3 - 9 z_1^2 z_2^3 + z_1 z_2^5 - 4 z_1 z_2^4 - 4 z_1 z_2 + 1$



What is an amoeba?

Example

1. $f := z_1 + z_2 + 1$
2. $f := z_1^3 z_2^3 - 9z_1^2 z_2^3 + z_1 z_2^5 - 4z_1 z_2^4 - 4z_1 z_2 + 1$



Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ is convex (Forsberg, Passare, Tsikh).

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ is convex (Forsberg, Passare, Tsikh).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ corresponds uniquely to a lattice point α in New(f) (via the ORDER MAP).

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ is convex (Forsberg, Passare, Tsikh).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ corresponds uniquely to a lattice point α in $\mathrm{New}(f)$ (via the ORDER MAP). Wether a complement component of $\mathcal{A}(f)$ exists depends on the choice of the coefficients of f.

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ is convex (Forsberg, Passare, Tsikh).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ corresponds uniquely to a lattice point α in $\text{New}(f)$ (via the ORDER MAP). Wether a complement component of $\mathcal{A}(f)$ exists depends on the choice of the coefficients of f. The complement components corresponding to the vertices of $\text{New}(f)$ do always exist (Forsberg, Passare, Tsikh).

Basic properties of amoebas

Let $f \in \mathbb{C}[\mathbf{z}]$ with amoeba $\mathcal{A}(f)$. Then:

- $\mathcal{A}(f)$ is a closed set with non–empty complement (Gelfand, Kapranov, Zelevinksy).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ is convex (Forsberg, Passare, Tsikh).
- Each complement component $E_\alpha(f)$ of $\mathcal{A}(f)$ corresponds uniquely to a lattice point α in New(f) (via the ORDER MAP). Wether a complement component of $\mathcal{A}(f)$ exists depends on the choice of the coefficients of f. The complement components corresponding to the vertices of New(f) do always exist (Forsberg, Passare, Tsikh).
- Every configuration of existing and non-existing inner complement components is possible (Rullgård).

The configuration space of amoebas

The configuration space of amoebas

- For $A \subset \mathbb{Z}^n$ the CONFIGURATION SPACE $\mathbb{C}_\diamond^A$ is the set

$$\left\{ f \in \mathbb{C}\left[\mathbf{z}^{\pm 1}\right] : f = \sum_{\alpha(j) \in A} b_j \cdot \mathbf{z}^{\alpha(j)}, b_j \in \mathbb{C}, \mathrm{New}(f) = \mathrm{conv}\, A \right\}$$

The configuration space of amoebas

- For $A \subset \mathbb{Z}^n$ the CONFIGURATION SPACE $\mathbb{C}_\diamond^A$ is the set

$$\left\{ f \in \mathbb{C}\left[\mathbf{z}^{\pm 1}\right] : f = \sum_{\alpha(j) \in A} b_j \cdot \mathbf{z}^{\alpha(j)}, b_j \in \mathbb{C}, \mathrm{New}(f) = \mathrm{conv}\, A \right\}$$

- For $\alpha \in A$ we define $U_\alpha^A \subseteq \mathbb{C}^A$ as the set

$$\left\{ f \in \mathbb{C}_\diamond^A : E_\alpha(f) \neq \emptyset \right\}.$$

The configuration space of amoebas

- For $A \subset \mathbb{Z}^n$ the CONFIGURATION SPACE $\mathbb{C}_\diamond^A$ is the set

$$\left\{ f \in \mathbb{C}\left[\mathbf{z}^{\pm 1}\right] : f = \sum_{\alpha(j) \in A} b_j \cdot \mathbf{z}^{\alpha(j)}, b_j \in \mathbb{C}, \mathrm{New}(f) = \mathrm{conv}\, A \right\}$$

- For $\alpha \in A$ we define $U_\alpha^A \subseteq \mathbb{C}^A$ as the set

$$\left\{ f \in \mathbb{C}_\diamond^A : E_\alpha(f) \neq \emptyset \right\}.$$

- Since every f in $\mathbb{C}_\diamond^A$ is determined by its coefficient vector, we may interpret $\mathbb{C}_\diamond^A$ as projective space with dimension $d = \#A$.

The configuration space of amoebas

Example

Let $f := b_0 + \sum_{j=1}^n b_j \cdot z_j$. Then

$$\mathbb{C}_\diamond^A = U_{\alpha(0)}^A = \cdots = U_{\alpha(n)}^A.$$

The configuration space of amoebas

Example

Let $f := b_0 + \sum_{j=1}^n b_j \cdot z_j$. Then

$$\mathbb{C}_\diamond^A = U_{\alpha(0)}^A = \cdots = U_{\alpha(n)}^A.$$

Unfortunately, this is the only example where all U_α^A are known ...

The configuration space of amoebas

Example

Let $f := b_0 + \sum_{j=1}^n b_j \cdot z_j$. Then

$$\mathbb{C}_\diamond^A = U_{\alpha(0)}^A = \cdots = U_{\alpha(n)}^A.$$

Unfortunately, this is the only example where all U_α^A are known ...
But:

Theorem (Rullgård)

The configuration space of amoebas

Example

Let $f := b_0 + \sum_{j=1}^n b_j \cdot z_j$. Then

$$\mathbb{C}_\diamond^A = U_{\alpha(0)}^A = \cdots = U_{\alpha(n)}^A.$$

Unfortunately, this is the only example where all U_α^A are known ...
But:

Theorem (Rullgård)

All U_α^A are non–empty, pseudo–convex, semialgebraic sets. The intersection of $(U_\alpha^A)^c$ with an arbitrary projective line in $\mathbb{C}_\diamond^A$ is non–empty and connected.

Key problems on the configuration spaces of amoebas

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^A$:

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?
- Are the U_α^A connected? ("*Jackpot question*")

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_{\diamond}^{\mathbf{A}}$:

- What is the geometric structure of the U_{α}^{A}?
- How can the U_{α}^{A} be computed?
- Are the U_{α}^{A} connected? ("*Jackpot question*")

MESSAGE OF THE TALK:

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_{\diamond}^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?
- Are the U_α^A connected? ("*Jackpot question*")

MESSAGE OF THE TALK:

Let $A := \{\alpha(0), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\alpha(1), \ldots, \alpha(n+1)$ span up a simplex with barycenter $\alpha(0)$. Then:

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}^{\mathbf{A}}_{\diamond}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?
- Are the U_α^A connected? ("*Jackpot question*")

MESSAGE OF THE TALK:

Let $A := \{\alpha(0), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\alpha(1), \ldots, \alpha(n+1)$ span up a simplex with barycenter $\alpha(0)$. Then:

- $U_{\alpha(0)}^A$ locally has the structure of an area bounded by a hypocycloid.

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?
- Are the U_α^A connected? (“*Jackpot question*”)

MESSAGE OF THE TALK:

Let $A := \{\alpha(0), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\alpha(1), \ldots, \alpha(n+1)$ span up a simplex with barycenter $\alpha(0)$. Then:

- $U_{\alpha(0)}^A$ locally has the structure of an area bounded by a hypocycloid.
- Membership of points in $U_{\alpha(j)}^A$ can be decided for every j.

Key problems on the configuration spaces of amoebas

KEY PROBLEMS ON $\mathbb{C}_\diamond^{\mathbf{A}}$:

- What is the geometric structure of the U_α^A?
- How can the U_α^A be computed?
- Are the U_α^A connected? ("*Jackpot question*")

MESSAGE OF THE TALK:

Let $A := \{\alpha(0), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\alpha(1), \ldots, \alpha(n+1)$ span up a simplex with barycenter $\alpha(0)$. Then:

- $U_{\alpha(0)}^A$ locally has the structure of an area bounded by a hypocycloid.
- Membership of points in $U_{\alpha(j)}^A$ can be decided for every j.
- All $U_{\alpha(j)}^A$ are pathconnected.

Complement components of amoebas for $\mathcal{P}_\Delta^*$

Initial Example

Initial Example

Example (Passare, Rullgård)

Let $f := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$.

Initial Example

Example (Passare, Rullgård)

Let $f := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$. Then $\mathcal{A}(f)$ is solid if and only if $\mathcal{A}(f)$ contains the origin which is the case if and only if

$$b_0 \notin K_n := \{-t_0 - \cdots - t_n : t_j \in \mathbb{C}, |t_j| = 1, t_0 \cdots t_n = 1\}.$$

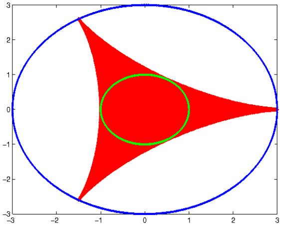
Initial Example

Example (Passare, Rullgård)

Let $f := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$. Then $\mathcal{A}(f)$ is solid if and only if $\mathcal{A}(f)$ contains the origin which is the case if and only if

$$b_0 \notin K_n := \{-t_0 - \cdots - t_n : t_j \in \mathbb{C}, |t_j| = 1, t_0 \cdots t_n = 1\}.$$

K_2 looks approximately the following way:



The class of investigated polynomials

The class of investigated polynomials

Definition

The class of investigated polynomials

Definition

1. Let $\mathcal{P}_\Delta$ be the class $f := b_0 + \sum_{j=1}^{n+1} b_i \cdot \mathbf{z}^{\alpha(j)}$ of Laurent–polynomials with $\mathrm{conv}\{\alpha(1), \ldots, \alpha(n+1)\}$ being a simplex containing the origin in the interior.

The class of investigated polynomials

Definition

1. Let $\mathcal{P}_\Delta$ be the class $f := b_0 + \sum_{j=1}^{n+1} b_i \cdot \mathbf{z}^{\alpha(j)}$ of Laurent–polynomials with $\mathrm{conv}\{\alpha(1), \ldots, \alpha(n+1)\}$ being a simplex containing the origin in the interior.
2. Let $\mathcal{P}_\Delta^* \subset \mathcal{P}_\Delta$ be the subclass with $\alpha(1) = -\sum_{j=2}^{n+1} \alpha(j)$.

The equilibrium point

The equilibrium point

Definition

The equilibrium point

Definition

Let $f \in \mathcal{P}_\Delta$. We define the EQUILIBRIUM POINT $\mathrm{eq}(f) \subset \mathbb{R}^n$ as the point with

$$\left|b_1 \cdot (\mathrm{Log}^{-1}(\mathrm{eq}(f)))^{\alpha(1)}\right| = \cdots = \left|b_{n+1} \cdot (\mathrm{Log}^{-1}(\mathrm{eq}(f)))^{\alpha(n+1)}\right|.$$

The equilibrium point

Definition

Let $f \in \mathcal{P}_\Delta$. We define the EQUILIBRIUM POINT $\mathrm{eq}(f) \subset \mathbb{R}^n$ as the point with

$$|b_1 \cdot (\mathrm{Log}^{-1}(\mathrm{eq}(f)))^{\alpha(1)}| = \cdots = |b_{n+1} \cdot (\mathrm{Log}^{-1}(\mathrm{eq}(f)))^{\alpha(n+1)}|.$$

Remark

$\mathrm{eq}(f)$ is the unique vertex of the tropical variety $\mathcal{T}(\mathrm{Trop}(f - b_0))$ obtained by "naive" tropicalization via Log–valuation

$$\mathrm{Trop}(f - b_0) := \bigoplus_{j=1}^{n+1} \log|b_j| \odot \langle \mathrm{Log}(\mathbf{z}), \alpha(j) \rangle$$

Initial example II

Initial example II

Example

Let $f_{b_0} := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^{n} z_j^{n+1}$.

Initial example II

Example

Let $f_{b_0} := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$. After rescaling by $f \to f \cdot \mathbf{z}^{-(1,\ldots,1)}$ we have $f \in \mathcal{P}_\Delta^*$.

Initial example II

Example

Let $f_{b_0} := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$. After rescaling by $f \to f \cdot \mathbf{z}^{-(1,\ldots,1)}$ we have $f \in \mathcal{P}_\Delta^*$. $eq(f)$ is the origin, since

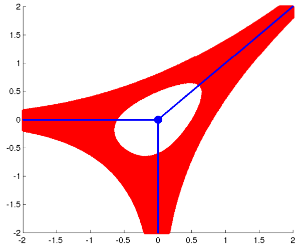
$$\left|\mathrm{Log}^{-1}(\mathbf{0})^{-(1,\ldots,1)}\right| = \left|\mathrm{Log}^{-1}(\mathbf{0})^{(0,\ldots,0,n+1,0,\ldots,0)}\right| = 1.$$

Initial example II

Example

Let $f_{b_0} := 1 + b_0 \cdot z_1 \cdots z_n + \sum_{j=1}^n z_j^{n+1}$. After rescaling by $f \to f \cdot \mathbf{z}^{-(1,\ldots,1)}$ we have $f \in \mathcal{P}_\Delta^*$. $eq(f)$ is the origin, since

$$\left|\mathrm{Log}^{-1}(\mathbf{0})^{-(1,\ldots,1)}\right| = \left|\mathrm{Log}^{-1}(\mathbf{0})^{(0,\ldots,0,n+1,0,\ldots,0)}\right| = 1.$$



The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$. Then the following statements are equivalent:

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$. Then the following statements are equivalent:

1. $f_{b_0} \in U_{\mathbf{0}}^A$ (i.e. $\mathcal{A}(f_{b_0})$ has genus 1),

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$. Then the following statements are equivalent:

1. $f_{b_0} \in U_{\mathbf{0}}^A$ (i.e. $\mathcal{A}(f_{b_0})$ has genus 1),
2. $\mathrm{eq}(f) \in E_{\mathbf{0}}(f_{b_0})$,

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$. Then the following statements are equivalent:

1. $f_{b_0} \in U_{\mathbf{0}}^A$ (i.e. $\mathcal{A}(f_{b_0})$ has genus 1),
2. $\mathrm{eq}(f) \in E_{\mathbf{0}}(f_{b_0})$,
3. $b_0 \notin \left\{-|\Theta| \cdot \sum_{j=0}^n e^{i\cdot(\arg(b_j)+\langle\alpha(j),\phi\rangle)} : \phi \in [0,2\pi)^n\right\}$.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- For every $\mathbf{w} \in \mathbb{R}^n$ there is a fiberfunction

$$\mathcal{F}_{\mathbf{w},f_{b_0}} : [0, 2\pi)^n \to \mathbb{C}, \phi \mapsto f(\mathrm{Log}^{-1}(\mathbf{w}), \phi).$$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- For every $\mathbf{w} \in \mathbb{R}^n$ there is a fiberfunction
$$\mathcal{F}_{\mathbf{w},f_{b_0}} : [0, 2\pi)^n \to \mathbb{C}, \phi \mapsto f(\mathrm{Log}^{-1}(\mathbf{w}), \phi).$$
- Let $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0})$ with $\mathbf{w} \neq \mathrm{eq}(f)$.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- For every $\mathbf{w} \in \mathbb{R}^n$ there is a fiberfunction

$$\mathcal{F}_{\mathbf{w},f_{b_0}} : [0,2\pi)^n \to \mathbb{C}, \phi \mapsto f(\mathrm{Log}^{-1}(\mathbf{w}), \phi).$$

- Let $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0})$ with $\mathbf{w} \neq \mathrm{eq}(f)$.
- Investigate the tropical hypersurface $\mathcal{T}$ given by

$$\bigoplus_{j=1}^{n+1} \log|b_j| \odot \langle \mathrm{Log}(\mathbf{z}), \alpha(j) \rangle$$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- For every $\mathbf{w} \in \mathbb{R}^n$ there is a fiberfunction
$$\mathcal{F}_{\mathbf{w},f_{b_0}} : [0, 2\pi)^n \to \mathbb{C}, \phi \mapsto f(\mathrm{Log}^{-1}(\mathbf{w}), \phi).$$
- Let $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0})$ with $\mathbf{w} \neq \mathrm{eq}(f)$.
- Investigate the tropical hypersurface $\mathcal{T}$ given by
$$\bigoplus_{j=1}^{n+1} \log|b_j| \odot \langle \mathrm{Log}(\mathbf{z}), \alpha(j)\rangle$$
Let $A_0, \ldots, A_n$ be the regions of $\mathbb{R}^n \setminus \mathcal{T}$ and let w.l.o.g. $\mathbf{w} \in A_0$.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- For every $\mathbf{w} \in \mathbb{R}^n$ there is a fiberfunction

$$\mathcal{F}_{\mathbf{w},f_{b_0}} : [0,2\pi)^n \to \mathbb{C}, \phi \mapsto f(\mathrm{Log}^{-1}(\mathbf{w}),\phi).$$

- Let $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0})$ with $\mathbf{w} \neq \mathrm{eq}(f)$.
- Investigate the tropical hypersurface $\mathcal{T}$ given by

$$\bigoplus_{j=1}^{n+1} \log|b_j| \odot \langle \mathrm{Log}(\mathbf{z}), \alpha(j)\rangle$$

  Let $A_0,\ldots,A_n$ be the regions of $\mathbb{R}^n \backslash \mathcal{T}$ and let w.l.o.g. $\mathbf{w} \in A_0$.
- **Key step**: For every $A_j \neq A_0$ there exists a $\mathbf{w}(j) \in \mathbb{R}^n$ such that there is an isomorphism $\pi_{\mathbf{w}(j)}$ on $[0,2\pi)^n$ with

$$\mathcal{F}_{\mathbf{w},f_{b_0}}(\phi) = \mathcal{F}_{\mathbf{w}(j),f_{b_0}}(\pi_{\mathbf{w}(j)}(\phi)).$$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- **Key step**: For every $A_j \neq A_0$ there exists a $\mathbf{w}(j) \in \mathbb{R}^n$ such that there is an isomorphism $\pi_{\mathbf{w}(j)}$ on $[0, 2\pi)^n$ with

$$\mathcal{F}_{\mathbf{w}, f_{b_0}}(\phi) \; = \; \mathcal{F}_{\mathbf{w}(j), f_{b_0}}(\pi_{\mathbf{w}(j)}(\phi)).$$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- **Key step**: For every $A_j \neq A_0$ there exists a $\mathbf{w}(j) \in \mathbb{R}^n$ such that there is an isomorphism $\pi_{\mathbf{w}(j)}$ on $[0, 2\pi)^n$ with

$$\mathcal{F}_{\mathbf{w},f_{b_0}}(\phi) \; = \; \mathcal{F}_{\mathbf{w}(j),f_{b_0}}(\pi_{\mathbf{w}(j)}(\phi)).$$

- Thus, in particular $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathbf{w}(j) \in E_{\mathbf{0}}(f_{b_0})$ for all j.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- **Key step**: For every $A_j \neq A_0$ there exists a $\mathbf{w}(j) \in \mathbb{R}^n$ such that there is an isomorphism $\pi_{\mathbf{w}(j)}$ on $[0, 2\pi)^n$ with

$$\mathcal{F}_{\mathbf{w}, f_{b_0}}(\phi) \; = \; \mathcal{F}_{\mathbf{w}(j), f_{b_0}}(\pi_{\mathbf{w}(j)}(\phi)).$$

- Thus, in particular $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathbf{w}(j) \in E_{\mathbf{0}}(f_{b_0})$ for all j.
- Hence, $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathrm{conv}\{\mathbf{w}, \mathbf{w}(1), \ldots, \mathbf{w}(n)\} \subset E_{\mathbf{0}}(f_{b_0})$ since $E_{\mathbf{0}}(f_{b_0})$ is convex.

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Sketch of proof for (1) $\Rightarrow$ (2):

- **Key step**: For every $A_j \neq A_0$ there exists a $\mathbf{w}(j) \in \mathbb{R}^n$ such that there is an isomorphism $\pi_{\mathbf{w}(j)}$ on $[0, 2\pi)^n$ with

$$\mathcal{F}_{\mathbf{w}, f_{b_0}}(\phi) \;=\; \mathcal{F}_{\mathbf{w}(j), f_{b_0}}(\pi_{\mathbf{w}(j)}(\phi)).$$

- Thus, in particular $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathbf{w}(j) \in E_{\mathbf{0}}(f_{b_0})$ for all j.
- Hence, $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathrm{conv}\{\mathbf{w}, \mathbf{w}(1), \ldots, \mathbf{w}(n)\} \subset E_{\mathbf{0}}(f_{b_0})$ since $E_{\mathbf{0}}(f_{b_0})$ is convex.
- Therefore, $\mathbf{w} \in E_{\mathbf{0}}(f_{b_0}) \Rightarrow \mathrm{eq}(f) \in E_{\mathbf{0}}(f_{b_0})$ since $\mathrm{eq}(f)$ is unique vertex of $\mathcal{T}$.

$\square$

The inner complement component of $\mathcal{A}(f)$ with $f \in \mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let f_{b_0} be a family of parametric polynomials in $\mathcal{P}_\Delta^*$ with parameter $b_0 \in \mathbb{C}$. Then the following statements are equivalent:

1. $f_{b_0} \in U_{\mathbf{0}}^A$ (i.e. $\mathcal{A}(f_{b_0})$ has genus 1),
2. $\mathrm{eq}(f) \in E_{\mathbf{0}}(f_{b_0})$,
3. $b_0 \notin \left\{-|\Theta| \cdot \sum_{j=0}^n e^{i\cdot(\arg(b_j)+\langle\alpha(j),\phi\rangle)} : \phi \in [0,2\pi)^n\right\}$.

Aim: Describe the geometric struture of the set
$S := \left\{b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(f),f_{b_0}}) \neq \emptyset\right\}$

Geometrical and topological structure of $(U_{\mathbf{0}}^A)^c$

Hypocycloids

Hypocycloids

Definition

Hypocycloids

Definition

For $R > r$, a HYPOCYCLOID with parameters $R, r \in \mathbb{R}_{>0}$ is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$(R - r) \cdot e^{i \cdot \phi} + r \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}, \quad \phi \in [0, 2\pi).$$

Hypocycloids

Definition

For $R > r$, a HYPOCYCLOID with parameters $R, r \in \mathbb{R}_{>0}$ is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$(R - r) \cdot e^{i \cdot \phi} + r \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}, \quad \phi \in [0, 2\pi).$$

Geometrically, it is the trajectory of some fixed point on a circle with radius r rolling (from the interior) on a circle with radius R.

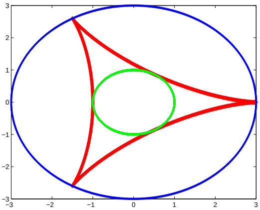
Hypocycloids

Definition

For $R > r$, a HYPOCYCLOID with parameters $R, r \in \mathbb{R}_{>0}$ is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$(R - r) \cdot e^{i \cdot \phi} + r \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}, \quad \phi \in [0, 2\pi).$$

Geometrically, it is the trajectory of some fixed point on a circle with radius r rolling (from the interior) on a circle with radius R.



$R = 3, \ r = 1$

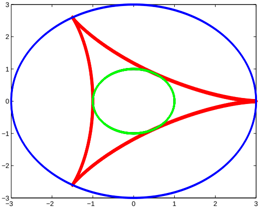
Hypocycloids

Definition

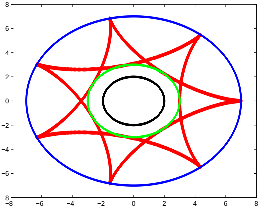
For $R > r$, a HYPOCYCLOID with parameters $R, r \in \mathbb{R}_{>0}$ is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$(R-r)\cdot e^{i\cdot\phi} + r\cdot e^{i\cdot\left(\frac{r-R}{r}\right)\cdot\phi}, \quad \phi \in [0, 2\pi).$$

Geometrically, it is the trajectory of some fixed point on a circle with radius r rolling (from the interior) on a circle with radius R.



$R = 3, \ r = 1$



$R = 7, \ r = 2$

Geometrical and topological structure of $(U_{\mathbf{0}}^A)^c$

Theorem (Theobald, dW.)

Let $A := \{\mathbf{0}, \alpha(1), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\operatorname{conv} A$ is a simplex with barycenter $\mathbf{0}$.

Geometrical and topological structure of $(U_{\mathbf{0}}^A)^c$

Theorem (Theobald, dW.)

Let $A := \{\mathbf{0}, \alpha(1), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\operatorname{conv} A$ is a simplex with barycenter $\mathbf{0}$. For $b_1, \ldots, b_{n+1} \in \mathbb{C}^*$ the intersection of $\partial(U_{\mathbf{0}}^A)^c$ with $h := \{(b_0 : b_1 : \cdots : b_{n+1}) : b_0 \in \mathbb{C}\}$ is the hypocycloid with parameters $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$ and cusps at

$$\arg(b_0) \quad = \quad \pi \cdot \left(1 + \frac{2k - \sum_{i=1}^{n} \arg(b_i)}{n+1}\right), \quad k \in \{0, \ldots, n\}.$$

Geometrical and topological structure of $(U_{\mathbf{0}}^A)^c$

Theorem (Theobald, dW.)

Let $A := \{\mathbf{0}, \alpha(1), \ldots, \alpha(n+1)\} \subset \mathbb{Z}^n$ such that $\operatorname{conv} A$ is a simplex with barycenter $\mathbf{0}$. For $b_1, \ldots, b_{n+1} \in \mathbb{C}^*$ the intersection of $\partial(U_{\mathbf{0}}^A)^c$ with $h := \{(b_0 : b_1 : \cdots : b_{n+1}) : b_0 \in \mathbb{C}\}$ is the hypocycloid with parameters $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$ and cusps at

$$\arg(b_0) \quad = \quad \pi \cdot \left(1 + \frac{2k - \sum_{i=1}^n \arg(b_i)}{n+1}\right), \quad k \in \{0, \ldots, n\}.$$

Furthermore, $(U_{\mathbf{0}}^A)^c$ is simply connected.

Example

Example

Example

We investigate the parametric family of polynomials

$$f_{b_0} \;=\; b_0 + z_1^{-1} z_2^{-3} + 2.4 \cdot z_1^1 z_2^{-2} + (1 + 1.3i) \cdot z_2^5.$$

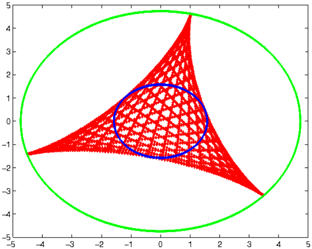
Example

Example

We investigate the parametric family of polynomials

$$f_{b_0} \;=\; b_0 + z_1^{-1} z_2^{-3} + 2.4 \cdot z_1^1 z_2^{-2} + (1 + 1.3i) \cdot z_2^5.$$

Then $\mathbb{C}_\diamond^A \cap \{(b_0 : 1 : 2.4 : 1 + 1.3 \cdot i) : b_0 \in \mathbb{C}\}$ looks approximately the following way:



Sketch of proof

- By former theorem: $\partial S = \partial (U_{\alpha(0)}^A)^c \cap h$ with
$$S = \left\{ b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(\mathbf{0}), f_{b_0}}) \neq \emptyset \right\}.$$

Sketch of proof

- By former theorem: $\partial S = \partial(U_{\alpha(0)}^A)^c \cap h$ with
$$S = \left\{ b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(\mathbf{0}),f_{b_0}}) \neq \emptyset \right\}.$$

- Let $k := -n + 1 + (-1)^{n+1}$ and
$$F : \ [k, n] \times [0, 2\pi) \to \mathbb{C},$$
$$(\mu, \psi) \mapsto |\Theta| \cdot \mu \cdot e^{i \cdot \psi} + |\Theta| \cdot e^{i \cdot (-n \cdot \psi + \sum_{j=1}^{n} \arg(b_j))}.$$

Sketch of proof

- By former theorem: $\partial S = \partial(U_{\alpha(0)}^A)^c \cap h$ with

$$S \;=\; \left\{b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(\mathbf{0}), f_{b_0}}) \neq \emptyset\right\}.$$

- Let $k := -n + 1 + (-1)^{n+1}$ and

$$F: \; [k, n] \times [0, 2\pi) \to \mathbb{C},$$
$$(\mu, \psi) \mapsto |\Theta| \cdot \mu \cdot e^{i \cdot \psi} + |\Theta| \cdot e^{i \cdot (-n \cdot \psi + \sum_{j=1}^{n} \arg(b_j))}.$$

1. The image of F is contained in S,

Sketch of proof

- By former theorem: $\partial S = \partial(U^A_{\alpha(0)})^c \cap h$ with
$$S \;=\; \left\{b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(\mathbf{0}),f_{b_0}}) \neq \emptyset\right\}.$$
- Let $k := -n + 1 + (-1)^{n+1}$ and
$$F: \;[k,n] \times [0,2\pi) \to \mathbb{C},$$
$$(\mu,\psi) \mapsto |\Theta| \cdot \mu \cdot e^{i\cdot\psi} + |\Theta| \cdot e^{i\cdot(-n\cdot\psi + \sum_{j=1}^{n} \arg(b_j))}.$$

1. The image of F is contained in S,
2. Up to a rotation, the curve parametrized by $\phi \mapsto F(n,\phi)$ for $\phi \in [0,2\pi)$ is a hypocyloid with $R = (n+1)\cdot|\Theta|$, $r = |\Theta|$.

Sketch of proof

- By former theorem: $\partial S = \partial(U_{\alpha(0)}^A)^c \cap h$ with

$$S \;=\; \left\{ b_0 \in \mathbb{C} : \mathcal{V}(\mathcal{F}_{\mathrm{eq}(\mathbf{0}), f_{b_0}}) \neq \emptyset \right\}.$$

- Let $k := -n + 1 + (-1)^{n+1}$ and

$$F: \; [k, n] \times [0, 2\pi) \to \mathbb{C},$$
$$(\mu, \psi) \mapsto |\Theta| \cdot \mu \cdot e^{i \cdot \psi} + |\Theta| \cdot e^{i \cdot (-n \cdot \psi + \sum_{j=1}^{n} \arg(b_j))}.$$

  1. The image of F is contained in S,
  2. Up to a rotation, the curve parametrized by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$ is a hypocyloid with $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$.

- Let T denote the region whose boundary is the hypocycloid given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. Then $S \subseteq T$ and $\partial T \subseteq \partial S$.

Sketch of proof

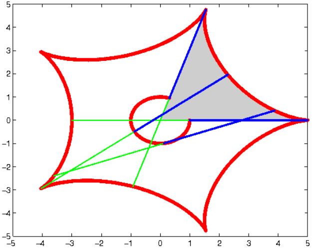
- Let T denote the region whose boundary is the hypocycloid given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. Then $S \subseteq T$ and $\partial T \subseteq \partial S$.

Sketch of proof

- Let T denote the region whose boundary is the hypocycloid given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. Then $S \subseteq T$ and $\partial T \subseteq \partial S$.
- The set S equals the region T whose boundary is the hypocycloid with parameter $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$ given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. In particular, S is simply connected.

Sketch of proof

- Let T denote the region whose boundary is the hypocycloid given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. Then $S \subseteq T$ and $\partial T \subseteq \partial S$.
- The set S equals the region T whose boundary is the hypocycloid with parameter $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$ given by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$. In particular, S is simply connected.



The Jackpot question for $\mathcal{P}_\Delta^*$

The Jackpot question for $\mathcal{P}_\Delta^*$

The Jackpot question for $\mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let $\mathbb{C}^A \subset \mathcal{P}_\Delta^*$. Then $U_{\mathbf{0}}^A$ is pathconnected.

The Jackpot question for $\mathcal{P}_\Delta^*$

Theorem (Theobald, dW.)

Let $\mathbb{C}^A \subset \mathcal{P}_\Delta^*$. Then $U_{\mathbf{0}}^A$ is pathconnected.

It suffices to show:

Theorem (Theobald, dW.)

Let $A := \{\alpha(1), \ldots, \alpha(d)\}$. If for every b the set

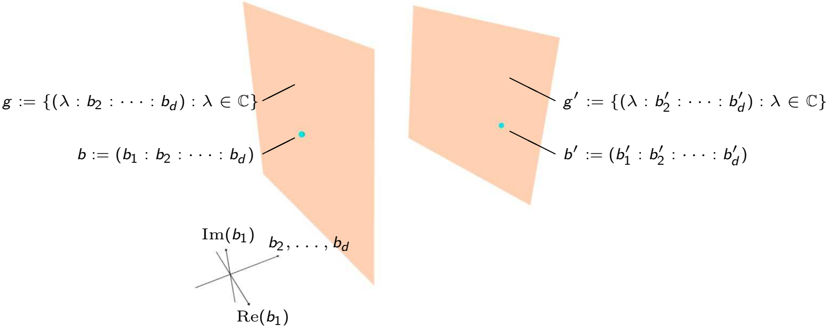
$$\{(\lambda \cdot b_1 : \cdots : b_d) : \lambda \in \mathbb{C}\} \cap \left(U_{\alpha(1)}^A\right)^c$$

is simply connected, then $U_{\alpha(1)}^A$ is pathconnected.

Proof

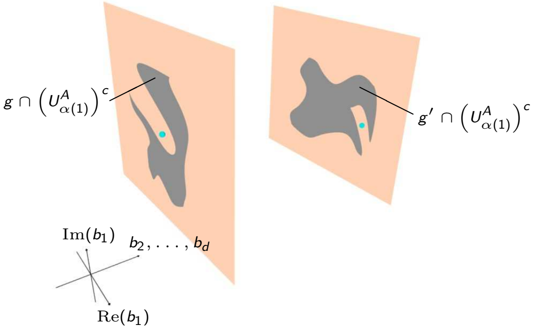
Proof

- Let $b, b' \in U_{\alpha(1)}^{A} \subset \mathbb{C}_{\diamond}^{A}$ in complex lines g, g'.



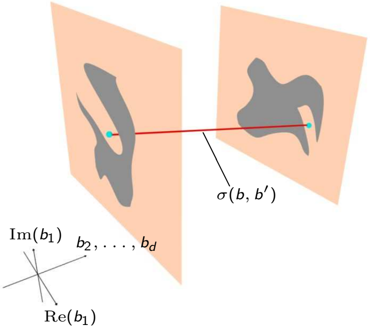
Proof

- $g \cap \left(U_{\alpha(1)}^A\right)^c$, $g' \cap \left(U_{\alpha(1)}^A\right)^c$ simply connected.



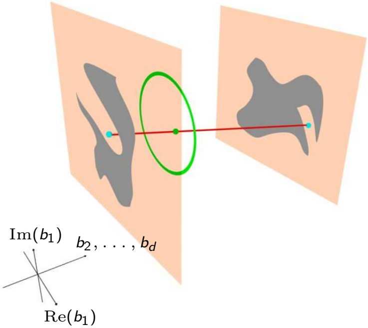
Proof

- Investigate line segment $\sigma(b, b') := b - \lambda \cdot (b' - b)$, $\lambda \in [0, 1]$.



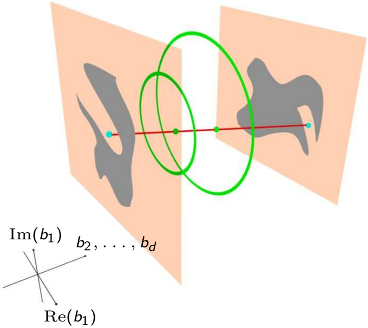
Proof

- $\forall p \in \sigma(b, b') \subset \mathbb{C}_\diamond^A$ compute minimal value of $|b_1|$, such that $\exists\, \mathbf{w} \in \mathbb{R}^n$ with $f_p\{\mathbf{w}\}$ being *lopsided* with dominating term $b_1 \cdot \mathbf{z}^{\alpha(1)}$.



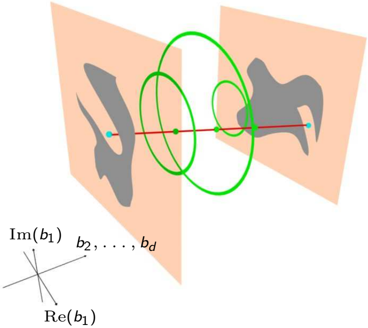
Proof

- $\forall p \in \sigma(b, b') \subset \mathbb{C}_\diamond^A$ compute minimal value of $|b_1|$, such that $\exists\, \mathbf{w} \in \mathbb{R}^n$ with $f_p\{\mathbf{w}\}$ being *lopsided* with dominating term $b_1 \cdot \mathbf{z}^{\alpha(1)}$.



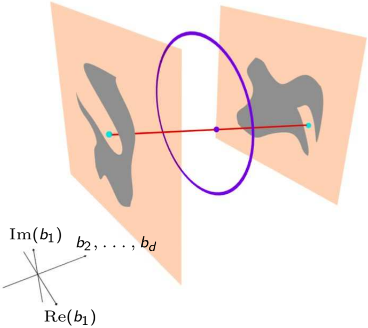
Proof

- $\forall p \in \sigma(b, b') \subset \mathbb{C}_\diamond^A$ compute minimal value of $|b_1|$, such that $\exists\, \mathbf{w} \in \mathbb{R}^n$ with $f_p\{\mathbf{w}\}$ being *lopsided* with dominating term $b_1 \cdot \mathbf{z}^{\alpha(1)}$.



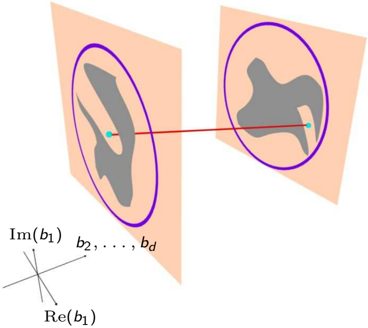
Proof

- Take the maximum $|\tilde{b_1}|$ of all these values.



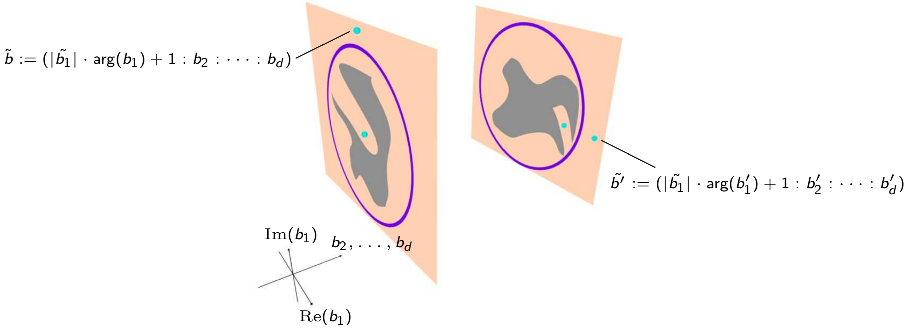
Proof

- Take the maximum $|\tilde{b_1}|$ of all these values.



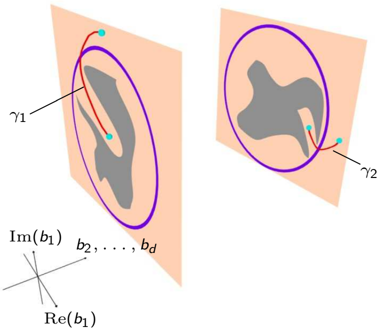
Proof

- Investigate the points $\tilde{b}, \tilde{b}'$;



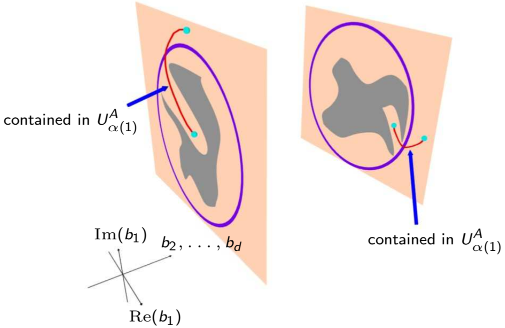
Proof

- Investigate the points $\tilde{b}, \tilde{b}'$; construct pathes γ_1 from b to $\tilde{b}$ and γ_2 from $\tilde{b}'$ to b'.



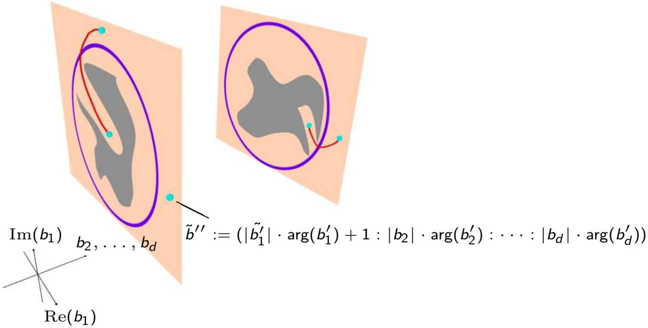
Proof

- γ_1 can be constructed in $U^A_{\alpha(1)}$ since $b, \tilde{b} \in U^A_{\alpha(1)}$ and $(U^A_{\alpha(1)})^c \cap g$ is simply connected (analogue for γ_2).



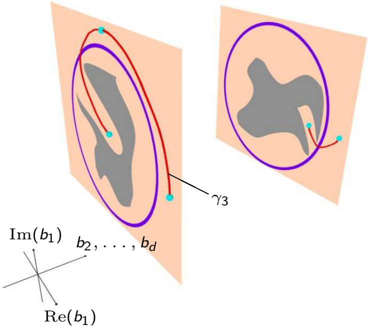
Proof

- Investigate $\tilde{b}''$;



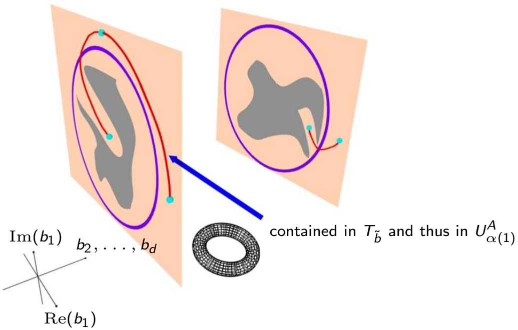
Proof

- Investigate $\tilde{b}''$; construct path γ_3 from $\tilde{b}$ to $\tilde{b}''$ in $T_{\tilde{b}} := \{(e^{i\cdot\phi_1} \cdot |\tilde{b_1}| \cdot \arg(b_1) : e^{i\cdot\phi_2} \cdot b_2 : \cdots : e^{i\cdot\phi_d} \cdot b_d) : \phi_1, \ldots, \phi_d \in [0, 2\pi)\}$.



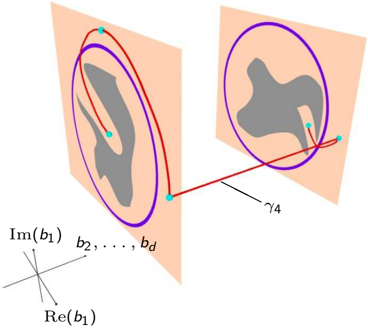
Proof

- $\gamma_3 \in U_{\alpha(1)}^A$ since: if $f_b\{\mathbf{w}\}$ is lopsided for some $\mathbf{w} \in \mathbb{R}^n$, then $f_{b'}\{\mathbf{w}\}$ is lopsided for every $b' \in T_{f_b}$.



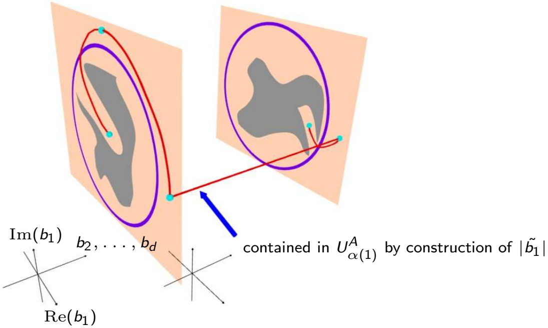
Proof

- Construct γ_4 as line segment $\tilde{b}'' + \lambda \cdot (\tilde{b}' - \tilde{b}'')$, $\lambda \in [0,1]$.



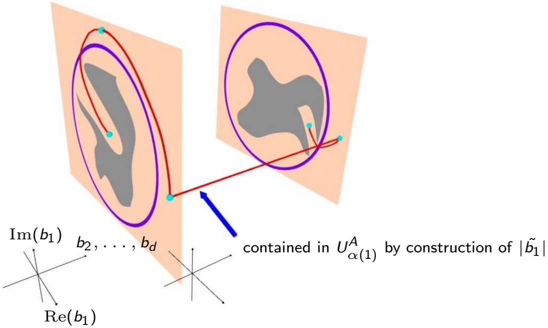
Proof

- γ_4 lies on affine line $\tilde{b}'' + \mu \cdot (0, |b_2| - |b_2|', \ldots, |b_d| - |b_d|')$. Every point of this line between $\tilde{b}''$ and $\tilde{b}'$ is lopsided due to construction of $|\tilde{b_1}|$. Hence, $\gamma_4 \in U_{\alpha(1)}^A$.



Proof

- Thus, $\gamma := \gamma_2 \circ \gamma_4 \circ \gamma_3 \circ \gamma_1$ is contained in $U^A_{\alpha(1)}$ and therefore b and b' are pathconnected.



Thank you for your attention!