The Configuration Space of Amoebas

Definition of Amoebas

Definition (Amoeba)**.** Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. Define:

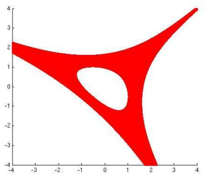
$$\mathrm{Log}: (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (|z_1| \cdot e^{i\cdot\phi_1}, \ldots, |z_n| \cdot e^{i\cdot\phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|)$$

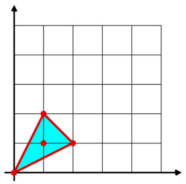
The AMOEBA $\mathcal{A}(f)$ of f is the image of $\mathcal{V}(f)$ under the Log-map.

- $\mathcal{A}(f)$ is a closed set with convex complement components $E_{\alpha(j)}(f)$.
- Each complement component $E_{\alpha(j)}(f)$ of $\mathcal{A}(f)$ corresponds uniquely to a lattice point $\alpha(j)$ in $\mathrm{New}(f)$ (via the ORDER MAP). Existence of certain $E_{\alpha(j)}(f)$ depends on the coefficients of f.
- Amoebas are connected to tropical hypersurfaces via SPINE and MASLOV DEQUANTIFIZATION.

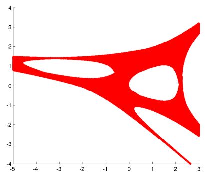
→ See e.g. *Forsberg, Gelfand, Kapranov, Mikhalkin, Passare, Rullgård, Tsikh, Zelevinsky et. al.*.

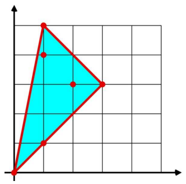
$f = z_1^2 z_2 + z_1 z_2^2 - 4 \cdot z_1 z_2 + 1$





$f = z_1^3 z_2^3 - 9 z_1^2 z_2^3 + z_1 z_2^5 - 4 z_1 z_2^4 - 4 z_1 z_2 + 1$





The Configuration Space of Amoebas

Definition (Configuration space)**.** For $A := \{\alpha(1), \ldots, \alpha(d)\} \subset \mathbb{Z}^n$ the CONFIGURATION SPACE $\mathbb{C}^A$ is

$$\mathbb{C}^A \quad := \quad \left\{ f = \sum_{i=1}^{d} b_i \mathbf{z}^{\alpha(i)} : \alpha(i) \in A, b_i \in \mathbb{C}, \mathrm{New}(f) = \mathrm{conv}(A) \right\}.$$

In $\mathbb{C}^A$ define for every $\alpha(j) \in \mathrm{conv}(A)$ the set

$$U_{\alpha(j)}^A \quad := \quad \left\{ f \in \mathbb{C}^A : E_{\alpha(j)}(f) \neq \emptyset \right\}$$

- Each $U_{\alpha(j)}^A$ is open, full dimensional and semi-algebraic.
- The complement of each $U_{\alpha(j)}^A$ is connected along every $\mathbb{C}$-line in $\mathbb{C}^A$.

KEY QUESTION: IS EVERY $U_{\alpha(j)}^A$ CONNECTED?

Minimally Sparse Polynomials

Definition (Minimally sparse)**.** A supportset $A \in \mathbb{Z}^n$ is called MINIMALLY SPARSE if $A = \mathrm{conv}(A) \cap \mathbb{Z}^n$.

Theorem. *Let $n = 1$, $A \subset \mathbb{Z}$ minimally sparse and $B \subseteq A$. Then $\bigcap_{\alpha(j)\in B} U_{\alpha(j)}^A$ is pathconnected.*

Conjecture. *The upper theorem holds for arbitrary $n \in \mathbb{N}$.*

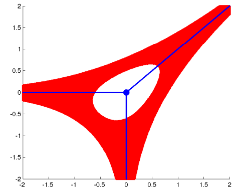
Polynomials with Barycentric Simplex Newton Polytope

Definition. A supportset $A = \{\alpha(0), \ldots, \alpha(n+1)\} \in \mathbb{Z}^n$ is called BARYCENTRIC WITH SIMPLEX NEWTON POLYTOPE if $\{\alpha(0), \ldots, \alpha(n)\}$ are the vertices of an n-simplex Δ and $\alpha(n+1)$ is the barycenter of Δ.

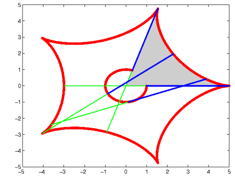
Theorem. *Let $n \geq 2$ and $A \subset \mathbb{Z}^n$ barycentric with simplex Newton polytope. Then*

- *If $\beta \in \mathrm{conv}(A) \setminus A$, then $U_\beta^A = \emptyset$.*
- *For all $b_1, \ldots, b_n \in \mathbb{C}^*$ the set $U_{\alpha(n+1)}^A \cap \{(1, b_1, \ldots, b_n, c) : c \in \mathbb{C}\}$ is pathconnected. Its complement is explicitly describeable.*
- *$U_{\alpha(n+1)}^A$ is pathconnected.*

Proof. Let $f = \sum_{\alpha(j)\in A} b_{\alpha(j)} \mathbf{z}^{\alpha(j)}$.



- For $A \in \mathbb{Z}^n$ barycentric with simplex Newton polytope the tropical hypersurface $\mathcal{T}(f)$ given by $\bigoplus_{\alpha(j)\in A\setminus\{\alpha(j):E_{\alpha(j)}(f)=\emptyset\}} b_{\alpha(j)} \odot \mathbf{z}^{\alpha(j)}$ is a deformation retract of $\mathcal{A}(f)$.
- If $f \in U_{\alpha(0)}$, then the unique vertex $\mathrm{eq}(f)$ of $\mathcal{T}(f - b_{\alpha(0)}\mathbf{z}^{\alpha(0)})$ is contained in $E_{\alpha(0)}(f)$.
- Let $\mathbb{F}_{\mathrm{eq}(f)}$ denote the fiber over $\mathrm{eq}(f)$ w.r.t. the Log–map. For all $b_1, \ldots, b_n \in \mathbb{C}^*$ $\mathbb{F}_{\mathrm{eq}(f)}$ intersects $\mathcal{V}(f)$ if and only if the coefficient $b_{n+1} \in \mathbb{C}$ is contained in a subset $S \subset \mathbb{C}$ bounded by the trajectory of a hypocycloid depending on the coefficients of f.
- The set S is simply connected.
- If for all $b_1, \ldots, b_n \in \mathbb{C}^*$ the set $(U_{\alpha(n+1)}^A)^c \cap \{(1, b_1, \ldots, b_n, c) : c \in \mathbb{C}\}$ is simply connected, then $U_{\alpha(n+1)}^A$ is pathconnected.

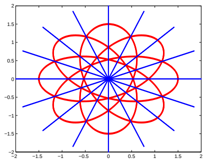


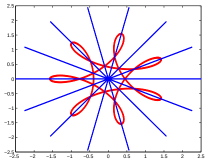
Trinomials

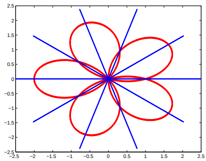
Let $f = z^s + p + qz^{-t}$ a trinomial with $p \in \mathbb{C}, q \in \mathbb{C}^*$. Modulis of such trinomials were e.g. described by *P. Bohl* in 1908. But the geometrical and topological structure of $\mathbb{C}^A$ is unknown so far.

Theorem. *f has a root of modulus $|z|$ if and only if p is located on the trajectory of a certain hypotrochoid curve depeding on s, t, q and $|z|$ (see also e.g. Neuwirth).*

Theorem. *If $U_j^A = \emptyset$, then p is located on the 1-fan $\{\lambda e^{i\cdot(s \arg(q) + k\pi/(s+t))} : \lambda \in \mathbb{R}_{>0}, k \in \{0, \ldots, 2(s+t)-1\}\}$. If $j \neq 0$, then $U_j^A \cap \{(1, p, q) : p \in \mathbb{C}\}$ is not connected.*

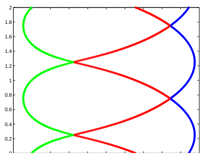


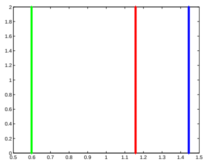




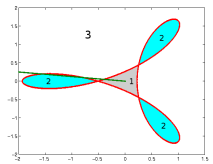
Example. *Let $f = z^2 + 1.5 \cdot e^{i\cdot\arg(p)} + e^{i\cdot\arg(q)} z^{-1}$.*

- AIM: *Construct a path γ in $\mathbb{C}^A$ from $(p_1, q_1) = (1.5 \cdot e^{i\cdot\pi/2}, 1)$ to $(p_2, q_2) = (1.5 \cdot e^{-i\cdot\pi/6}, 1)$ such that $\gamma \in U_1^A$.*
- *Impossible if $\arg(q) = 0$ for every point on γ.*
- *Possible along $\gamma : [0,1] \to \mathbb{C}^A, k \mapsto (1.5 \cdot e^{i(1/4+2k/3)\cdot\pi/2}, e^{i\cdot 2k\pi})$.*





Conjecture. *For trinomials the sets U_j^A are pathconnected but not simply connected in $\mathbb{C}^A$.*



Corollary. *For $f = \sum_{\alpha(j)\in A} b_{\alpha(j)} \mathbf{z}^{\alpha(j)}$ a complement component $E_{\alpha(j)}(f)$ is not monotonically growing in $|b_{\alpha(j)}|$ in general.*

- *$f_p = z^2 - |p| \cdot e^{i\cdot\varepsilon\pi} + z^{-1}$ is a counterexample for $\varepsilon > 0$ sufficiently small. The figure shows this for $|z| = |0.925|$.*

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see ArXiv 1108.2456.