Approximation of Amoebas and Coamoebas by Sums of Squares

Definition: SDPs

Definition. Let $C, A_1, \ldots, A_m$ be real, symmetric $n \times n$ matrices and $b_1, \ldots, b_m \in \mathbb{R}^m$. A SEMIDEFINITE OPTIMIZATION PROBLEM (SDP) is given by

$$\begin{aligned} &\inf \operatorname{Tr}(X, C) \text{ s.t.} \\ &\operatorname{Tr}(A_i, X) = b_i \text{ for } 1 \le i \le m \\ &X \succeq 0 \end{aligned}$$

where $X \succeq 0$ means that X is positive semidefinite.

Proposition. *Let $g \in \mathbb{R}[x_1, \ldots, x_n], \operatorname{tdeg}(g) = 2d$ and Y the vector of all monomials in $x_1, \ldots, x_n$ with degree $\le d$. g is **sum of squares (SOS)** iff there exists a matrix Q with $Q \succeq 0$ and*

$$g \quad = \quad Q^T Y Q.$$

Real Nullstellensatz

Proposition. *For polynomials $g_1, \ldots, g_r \in \mathbb{R}[\mathbf{X}]$ and $I := \langle g_1, \ldots, g_r \rangle \subset \mathbb{R}[\mathbf{X}]$ the following statements are equivalent:*

- *The real variety $\mathcal{V}_{\mathbb{R}}(I)$ is empty.*
- *There exist a polynomial $G \in I$ and a sum of squares polynomial H with*
$$G + H + 1 \;=\; 0\,.$$

Definition: Amoebas and Coamoebas

Definition. Let $f \in \mathbb{C}[\mathbf{Z}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ and

$$\operatorname{Log}: (\mathbb{C}^*)^n \to \mathbb{R}^n,$$
$$(|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|)$$
$$\operatorname{Arg}: (\mathbb{C}^*)^n \to [0, 2\pi)^n,$$
$$(|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\phi_1, \ldots, \phi_n)$$

The AMOEBA $\mathcal{A}(f)$ of f is the image of $\mathcal{V}(f)$ under the Log-map; the COAMOEBA $co\mathcal{A}(f)$ is the image of $\mathcal{V}(f)$ under the Arg-map .

- $\mathcal{A}(f)$ is a closed set with convex complement components $E_{\alpha(j)}(f)$.
- Each $E_{\alpha(j)}(f)$ corresponds uniquely to a lattice point $\alpha(j)$ in $\operatorname{New}(f)$ (via ORDER MAP).

→ For structural results on (co)amoebas see e.g. *Forsberg, Nilsson, Nisse, Passare, Purbhoo, Rullgård, Tsikh*

A Key Question: Membership

For given $f := \sum_{j=1}^{d} b_j \cdot \mathbf{z}^{\alpha(j)} \in \mathbb{C}[\mathbf{Z}]$ resp. $I \subseteq \mathbb{C}[\mathbf{Z}]$ **membership** is a key question on (co)amoebas:

Decide for $\lambda \in \mathbb{R}^n$ (resp. $\phi \in [0, 2\pi)^n$) wether $\lambda \in \mathcal{A}(I)$ (resp. $\phi \in co\mathcal{A}(I)$)

Amoeba Membership via Real Nullstellensatz

Let $f_k = \sum_{j=1}^{d_k} b_{k,j} \cdot \mathbf{z}^{\alpha(k,j)} \in \mathbb{C}[\mathbf{Z}]$ with $\alpha(k,1), \ldots, \alpha(k, d_k) \in \mathbb{N}_0^n$ spanning $\mathbb{R}^n$. For any $\lambda \in (0, \infty)^n$ set

$$\mu_{k,j} \;:=\; \lambda^{\alpha(k,j)} \;=\; \lambda_1^{\alpha(k,j)_1} \cdots \lambda_n^{\alpha(k,j)_n}\,, \quad 1 \le j \le d\,.$$

Define the monomials $m_{k,j} := \mathbf{z}^{\alpha(k,j)} = z_1^{\alpha(k,j)_1} \cdots z_n^{\alpha(k,j)_n}$.
Write every $f(\mathbf{z})$ as $f(\mathbf{x} + i \cdot \mathbf{y}) = f^{\mathrm{re}}(\mathbf{x}, \mathbf{y}) + i \cdot f^{\mathrm{im}}(\mathbf{x}, \mathbf{y})$ with $f^{\mathrm{re}}, f^{\mathrm{im}} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. Let $I := \langle f_1, \ldots, f_r \rangle$, and $I', I^* \subset \mathbb{R}[\mathbf{X}, \mathbf{Y}]$ be generated by

$$I' \;:=\; \{f_j^{\mathrm{re}}, f_j^{\mathrm{im}} : 1 \le j \le r\} \cup \{x_k^2 + y_k^2 = \lambda_j^2 : 1 \le j \le n\}$$
$$I^* \;:=\; \{f_j^{\mathrm{re}}, f_j^{\mathrm{im}} : 1 \le j \le r\} \cup \bigcup_{k=1}^{r} \{(m_{k,j}^{\mathrm{re}})^2 + (m_{k,j}^{\mathrm{im}})^2 - \mu_{k,j}^2 \,:\, 1 \le j \le d_k\}\,.$$

The UNLOG AMOEBA is defined as $\mathcal{U}(I) := \{(|z_1|, \ldots, |z_n|) : \mathbf{z} \in \mathcal{V}(I)\}$.

Polynomial based formulation

Either a point $(\lambda_1, \ldots, \lambda_n)$ is contained in $\mathcal{U}(I)$ or there exist a $G \in I'$ and and SOS $H \in \mathbb{R}[\mathbf{X}, \mathbf{Y}]$ with

$$G + H + 1 \quad = \quad 0\,.$$

Monomial based formulation

Either a point $(\lambda_1, \ldots, \lambda_n)$ is contained in $\mathcal{U}(I)$, or there exist polynomials $G \in I^*$ and an SOS $H \in \mathbb{R}[\mathbf{X}, \mathbf{Y}]$ with

$$G + H + 1 \quad = \quad 0\,.$$

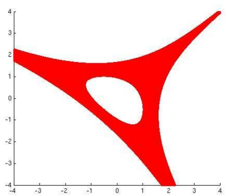
Coamoeba Membership via Real Nullstellensatz

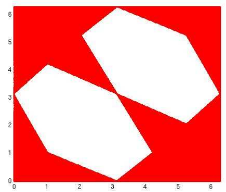
We have $0 \in co\mathcal{A}(I) \iff \{z = x + iy \in (\mathbb{C}^*)^n \,:\, z \in \mathcal{V}(I) \text{ and } x_k \ge 0,\ y_k = 0\ \forall k\} \neq \emptyset$. Replace "$x_k \ge 0$" by considering x_k^2 in arguments of $f_1, \ldots, f_r$. Then $0 \in \operatorname{complement}(co\mathcal{A}(I))$ iff there exists a polynomial identity

$$\sum_{i=1}^{r} c_i \cdot f_i(\mathbf{x}^2, \mathbf{y})^{\mathrm{re}} + \sum_{i=1}^{r} c_i' \cdot f_i(\mathbf{x}^2, \mathbf{y})^{\mathrm{im}} + \sum_{j=1}^{n} d_j \cdot y_j + H + 1 \;=\; 0$$

with $c_i, c_i', d_j \in \mathbb{R}[\mathbf{X}, \mathbf{Y}]$ and an SOS H.

Amoeba and coamoeba of
$f := z_1^2 z_2 + z_1 z_2^2 - 4 \cdot z_1 z_2 + 1$





Lopsidedness

Let $g \in \mathbb{C}[\mathbf{Z}]$ with $g(z) = \sum_{i=1}^{d} m_i(z)$. For $w \in \mathbb{R}^n$ define

$$g\{w\} \;:=\; \left(\left|m_1\left(\operatorname{Log}^{-1}(w)\right)\right|, \ldots, \left|m_d\left(\operatorname{Log}^{-1}(w)\right)\right|\right).$$

A list is called LOPSIDED if one of the numbers is greater than the sum of all the others. Define

$$\mathcal{LA}(g) \;:=\; \{w \in \mathbb{R}^n \;:\; g\{w\} \text{ is not lopsided}\}\,.$$

Note that $\mathcal{A}(g) \subseteq \mathcal{LA}(g)$. Define

$$\tilde{g}_k(z) \quad := \quad \prod_{j_1=0}^{k-1} \cdots \prod_{j_d=0}^{k-1} g\left(e^{2\pi i\, j_1/k} z_1, \ldots, e^{2\pi i\, j_d/k} z_n\right)$$

Theorem (Purbhoo 08). *For $k \to \infty$ the family $\mathcal{LA}(\tilde{g}_k)$ converges uniformly to $\mathcal{A}(g)$. $\mathcal{A}(g)$ can be approximated by $\mathcal{LA}(\tilde{g}_k)$ within an $\varepsilon > 0$ if k is greater than some $N_{(\varepsilon, g)} \in \mathbb{N}$.*

Main Results: Degree Bounds

Theorem. *If $w = \mathbf{1} \in \operatorname{complement}(\mathcal{U}(f))$ with $f\{w\}$ lopsided, then there exists a certificate $\sum_{i=1}^{d+2} s_i g_i + H + 1 = 0$ of total degree $2 \cdot \operatorname{tdeg}(f)$, where $g_1 = |b_1|^2$, $g_i = -|b_i| \cdot \sum_{k=2}^{d} |b_k|$, $2 \le i \le d$,*

$$g_{d+1} = \left(-b_1 \cdot z^{\alpha(1)} + \sum_{i=2}^{d} b_i \cdot z^{\alpha(i)}\right)^{\mathrm{re}}, \quad g_{d+2} = \left(-b_1 \cdot z^{\alpha(1)} + \sum_{i=2}^{d} b_i \cdot z^{\alpha(i)}\right)^{\mathrm{im}},$$

$$H = \sum_{2 \le i < j \le d} |b_i| \cdot |b_j| \cdot \left(\frac{b_i^{\mathrm{re}}}{|b_i|} \cdot \left(z^{\alpha(i)}\right)^{\mathrm{re}} - \frac{b_j^{\mathrm{re}}}{|b_j|} \cdot \left(z^{\alpha(j)}\right)^{\mathrm{re}}\right)^2 + |b_i| \cdot |b_j| \cdot \left(\frac{b_i^{\mathrm{im}}}{|b_i|} \cdot \left(z^{\alpha(i)}\right)^{\mathrm{im}} - \frac{b_j^{\mathrm{im}}}{|b_j|} \cdot \left(z^{\alpha(j)}\right)^{\mathrm{im}}\right)^2.$$

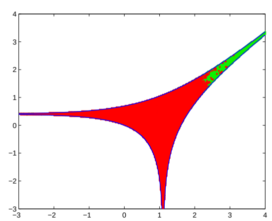
Corollary. *Let $r \in \mathbb{N}$.*

1. *For any $w \in \mathbb{R}^n \setminus \mathcal{LA}(\tilde{f}_k) \subset \mathbb{R}^n \setminus \mathcal{A}(f)$ there exists a certificate of degree at most $2 \cdot k^n \cdot \deg(f)$ which can be computed explicitly. In particular, for linear hyperplane amoebas in $\mathbb{R}^n$, any point in the complement of the amoeba has a certificate whose sum of squares is a sum of squares of affine functions.*
2. *The certificate determines the order of the complement component . to which w belongs.*

Actual Computations

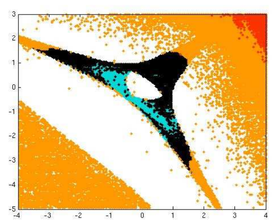
Using `SOSTools` with SDP solver `SeDuMi`

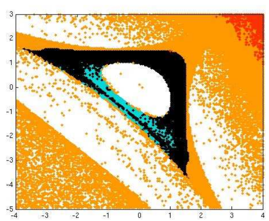
$f := z_1 + 2z_2 + 3$



white: SDP feasible, *red*: SDP infeasible, *green*: recognzied as infeasible, with numerical issues

$f := z_1^2 z_2 + z_1 z_2^2 + c \cdot z_1 z_2 + 1$ with $c = 2$ and $c = -4$





white: SDP feasible, *orange*: recognized as feasible, with numerical issues, *black*: SDP infeasible, *turquoise*: recognized as infeasible, with numerical issues; degree bound: 3

Thorsten Theobald
theobald@math.uni-frankfurt.de
http://www.math.uni-frankfurt.de/~theobald/

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Timo de Wolff
wolff@math.uni-frankfurt.de
http://www.math.uni-frankfurt.de/~wolff/

see ArXiv 1101.4114.