Amoebas of genus at most 1

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Introduction

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  with $\arg(c) \in [0, 2]$ and $|c|$ denoting the modulus.
- The NEWTON POLYTOPE of a polynomial f is the polytope obtained by taking the convex hull of all its exponent vectors interpreted as points in $\mathbb{Z}^n$ and will be denoted as $\mathrm{New}(f)$.

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Let $f := z_1^2 z_2^3 + 2z_1^2 z_2^2 + 4z_1 z_2^2 + z_1 z_2 + 3z_2^2 + 1$. Then we have

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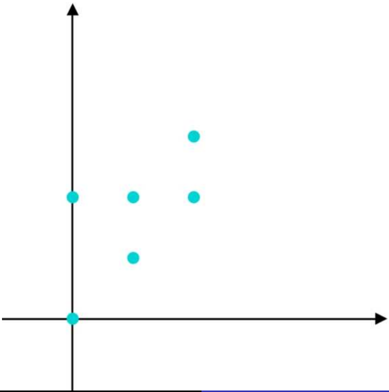

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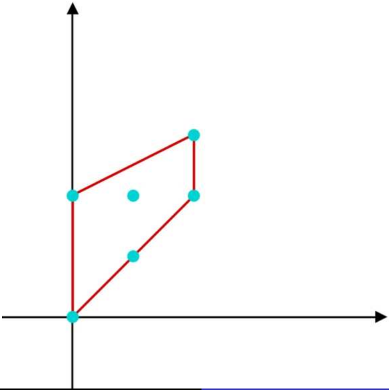

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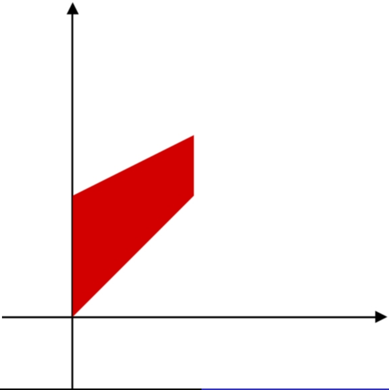

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$$\begin{aligned} \mathrm{Log}: \quad & (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \\ & (|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|) \end{aligned}$$

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Then the AMOEBA $\mathcal{A}(f)$ of f is the image of $\mathcal{V}(f)$ under the Log-map.

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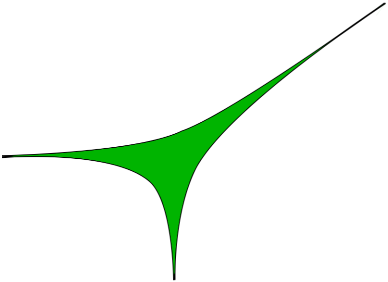

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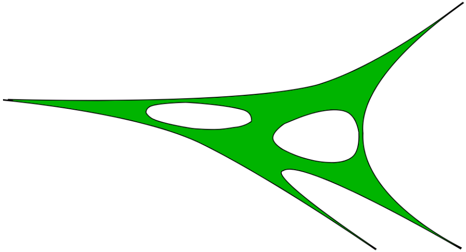

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- Every configuration of existing and non-existing inner complement components is possible (Rullgård).

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These two problems are hard in general but may be solved for a rich subclass of polynomials!

The spine and the amoebas equilibrium

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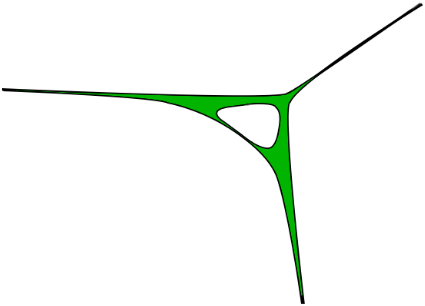

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Aim

Define a nice polyhedral complex structure on $\mathcal{A}(f)$ preserving the homotopy of $\mathcal{A}(f)$.

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- The Ronkin function maps every point **w** to the average log–value of f on the fibre of **w** under the Log–map.
- The Ronkin function is convex and piecewise linear on each complement component of $\mathcal{A}(f)$.

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Idea:

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3. take the subset of $\mathbb{R}^n$ where the maximum is attained at least twice.

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Definition (Passare, Rullgård)

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Then the SPINE $\mathcal{S}(f)$ of f is defined as the set of points where

$$S(\mathbf{w}) \quad := \quad \max_{x^{(i)} \in A} \left(\beta_{x^{(i)}} + \left\langle x^{(i)}, \mathbf{w} \right\rangle \right).$$

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- The spine is the zero locus of the tropical polynomial

$$\bigoplus_{x^{(i)} \in A} \log|\beta_{x^{(i)}}| \odot \left\langle \mathbf{w}, x^{(i)} \right\rangle$$

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- The Ronkin coefficients $\beta_{x(i)}$ may not be described in a combinatorial way out of the original polynomial f since they are given by

$$\beta_{x(i)} = \underbrace{\log|b_i|}_{\text{coefficient of } z^{x^{(i)}}} + \text{convergent laurent series}$$

The amoebas equilibrium

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Let $f := \sum_{i=1}^{m} b_i \cdot \mathbf{z}^{x^{(i)}} \in \mathbb{C}[\mathbf{z}]$ and A denote the lattice points whose corresponding compl. comp. in $\mathcal{A}(f)$ exists. Then we define:

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(a) The EQUILIBRIUM $\mathcal{E}(f)$ of f as the set of all $\mathbf{w} \in \mathrm{Log}\left((\mathbb{C}^*)^n\right)$ s.t. at least two monomials of f have the same modular value on $\mathrm{Log}^{-1}(\mathbf{w})$.

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(b) The AMOEBAS EQUILIBRIUM $\mathcal{V}(\operatorname{Trop}(f_{|A}))$ as the zero locus of

$$\operatorname{Trop}(f_{|A}) \quad := \quad \bigoplus_{i \in \{j \,|\, x^{(j)} \in A\}}^{n} \log|b_i| \odot \langle \mathbf{w}, x^{(i)} \rangle.$$

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- Both structures are polyhedral complexes.
- $\mathcal{V}(\mathrm{Trop}(f_{|A}))$ is dual to a regular integral subdivision of $\mathrm{New}(f)$.
- We call the vertices of $\mathcal{E}(f)$ where $n+1$ monomials have the same weight the EQUILIBRIUM POINTS of f.

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Example

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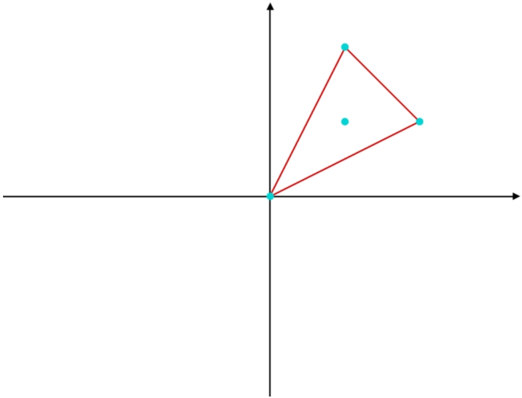

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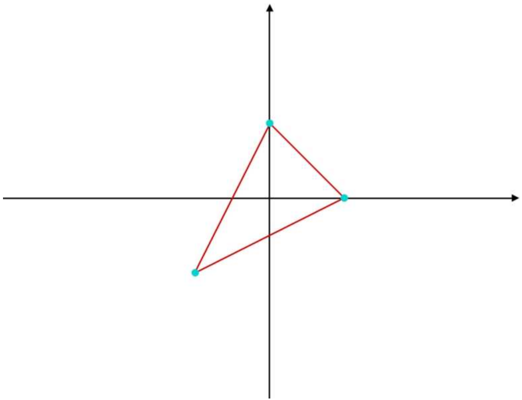

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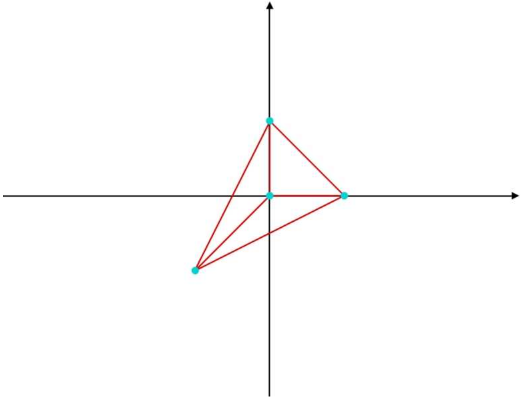

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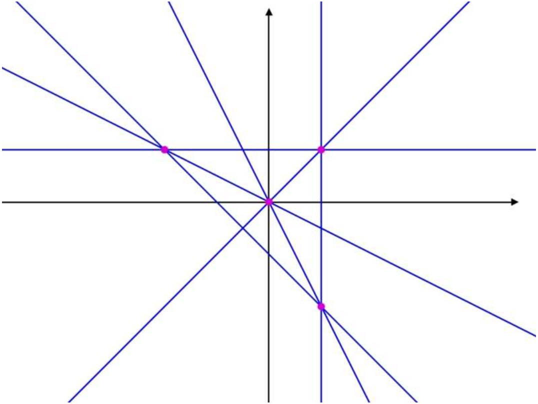

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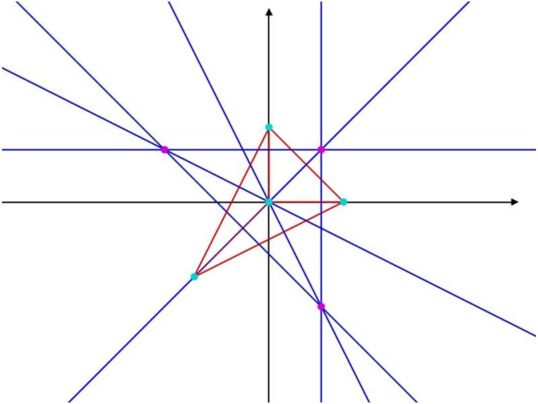

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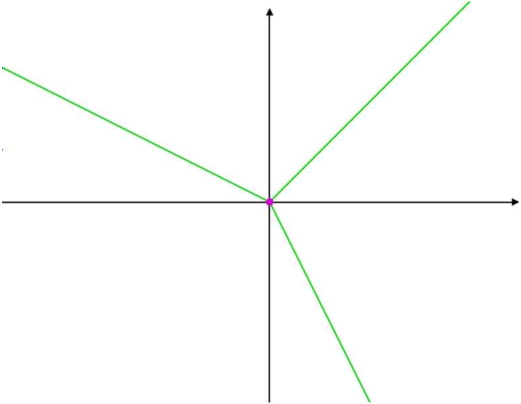

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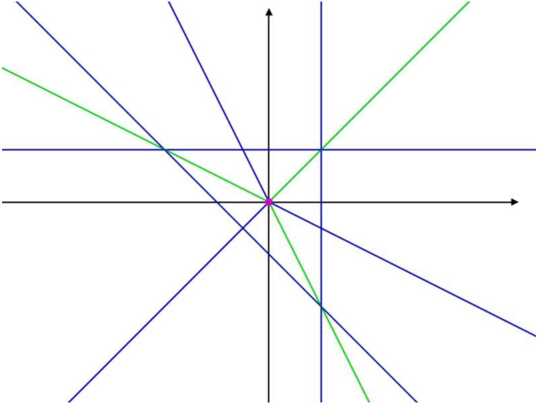

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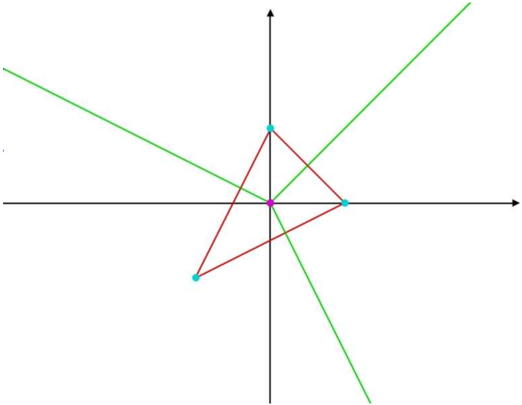

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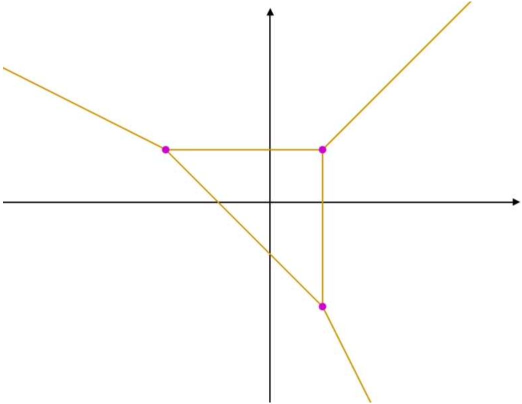

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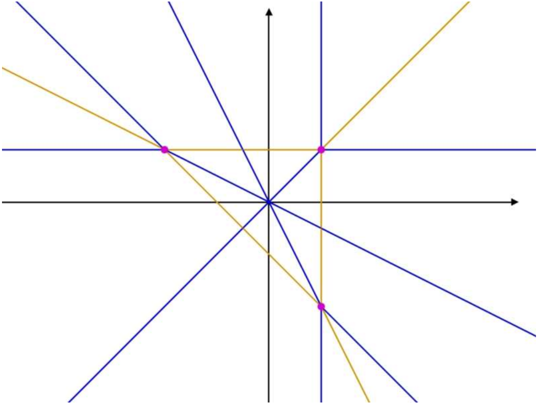

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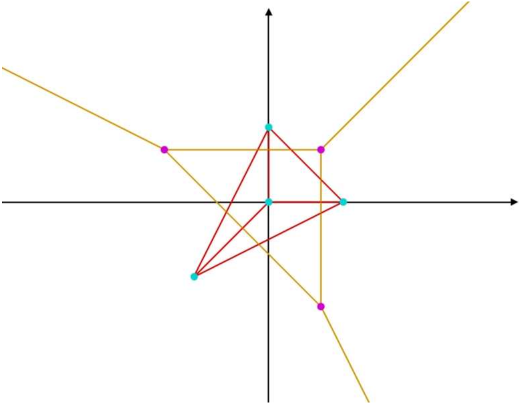

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W.l.o.g. we define $b_0 := 1, x^{(0)} := (0, \ldots, 0)^T$ and say f is $(\Delta, \cdot)$.

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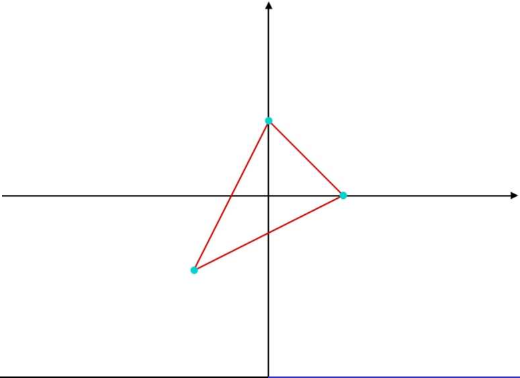

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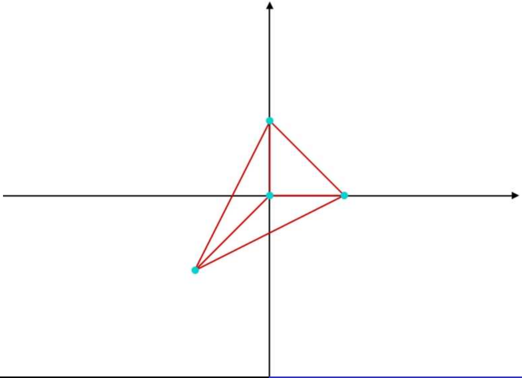

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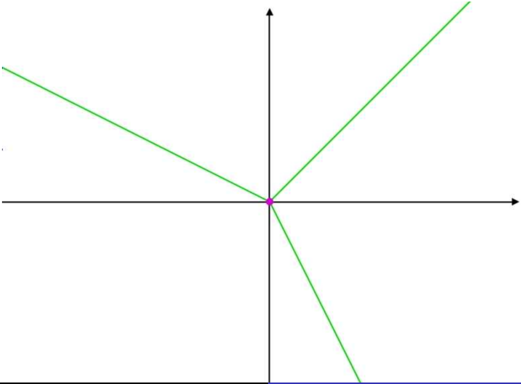

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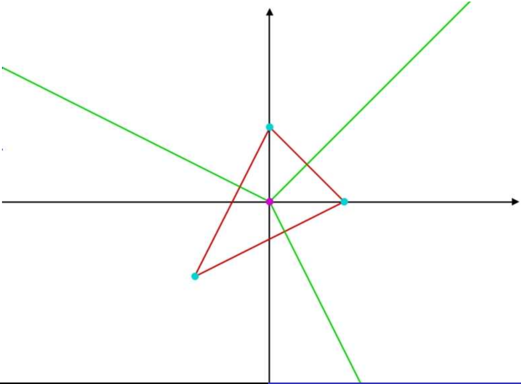

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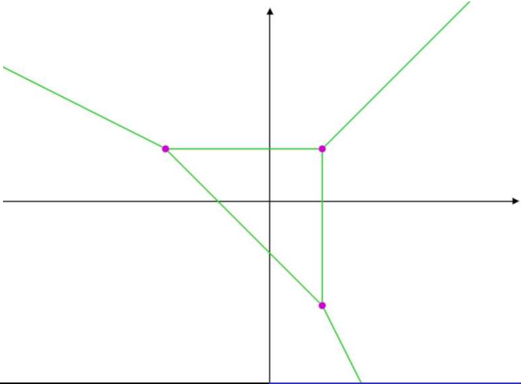

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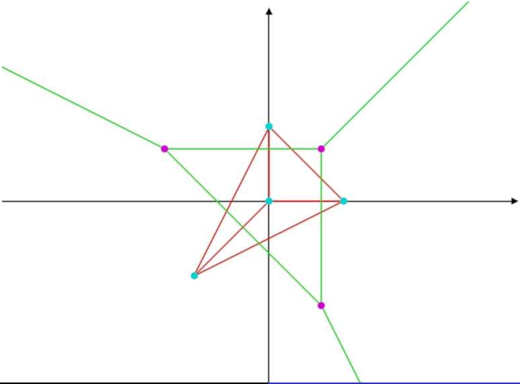

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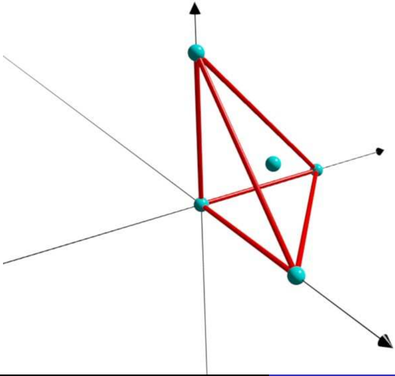

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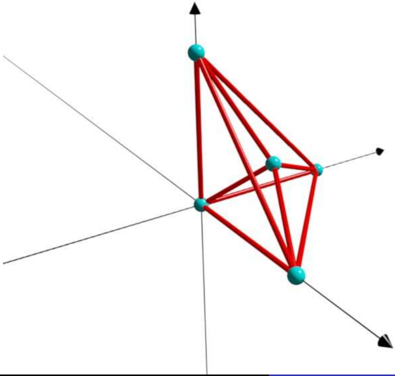

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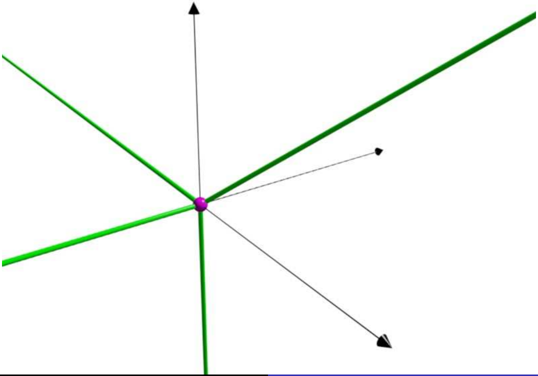

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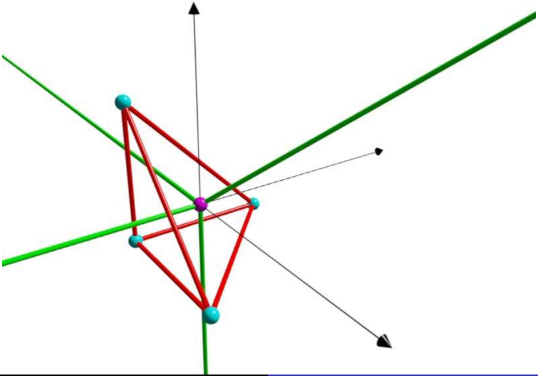

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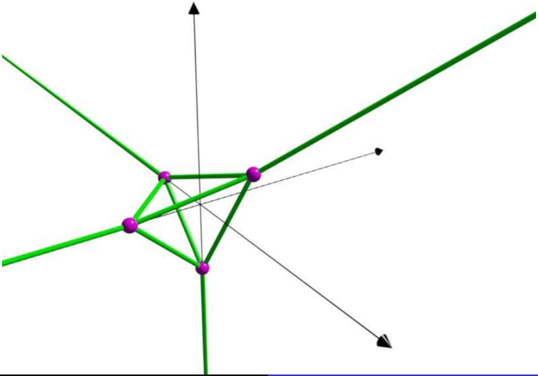

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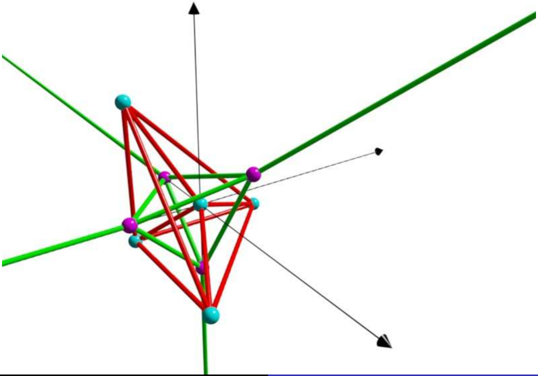

Boundaries for the inner complement component

Rough boundary theorem

Aim

Rough boundary theorem

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Let $b_1, \ldots, b_n$ be arbitrary. Find general boundaries for $|c|$ depending on $b_1, \ldots, b_n$ and $\arg(c)$ s.t. $\mathcal{A}(f)$ has genus 0 resp. genus 1.

Rough boundary theorem

Theorem (Theobald, dW.)

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Sketch of proof

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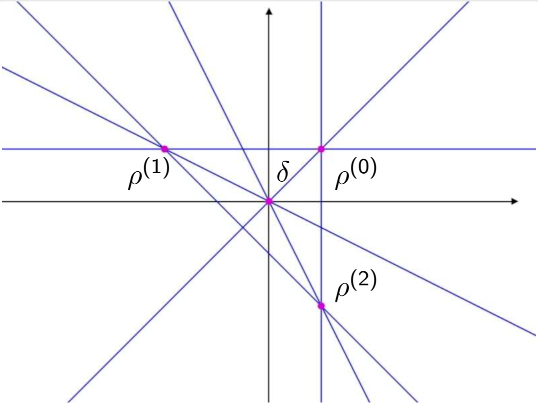

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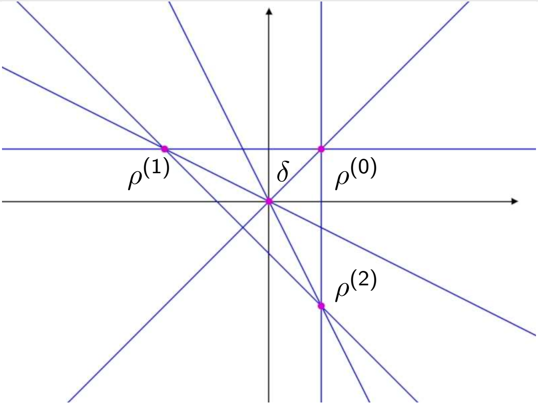

Lemma (Theobald, dW.)

All the equilibrium points $\delta, \rho^{(0)}, \ldots, \rho^{(n)}$ may be calculated in terms of $b_0, \ldots, b_n, c$ and the exponents of f.

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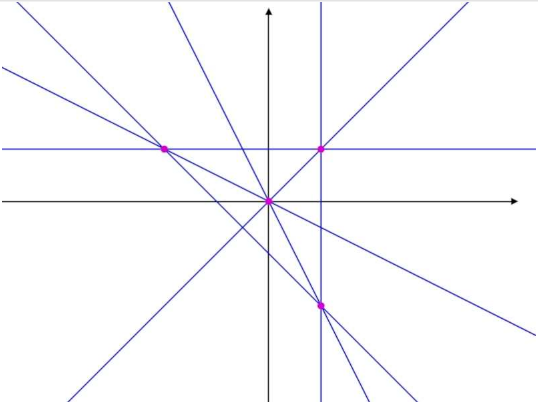

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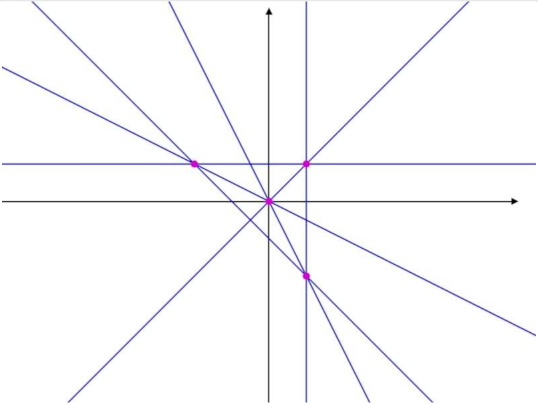

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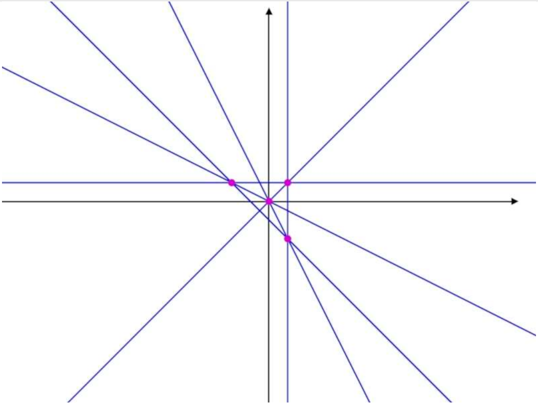

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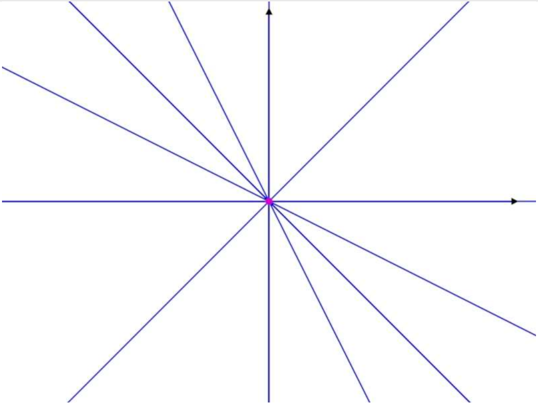

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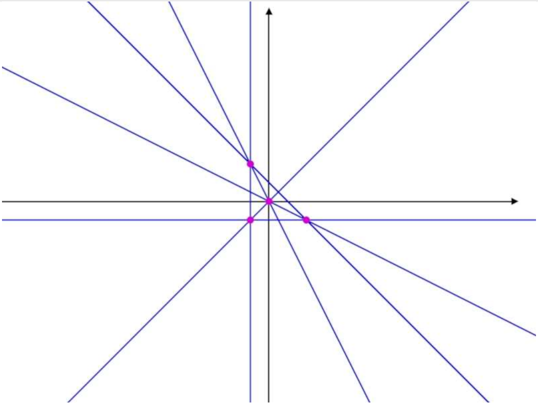

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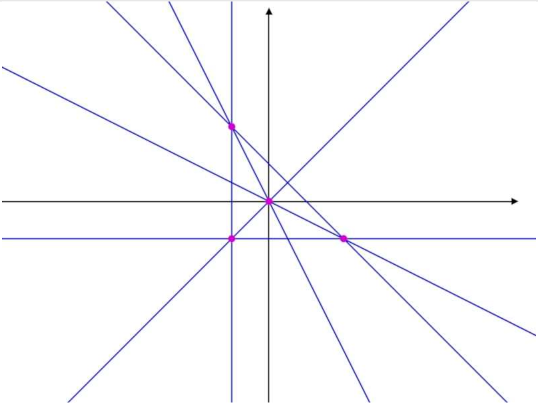

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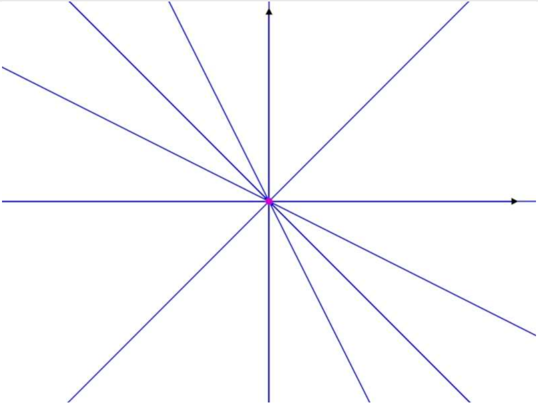

Observe:

There is a choice for $|c|$ s.t. $\delta = \rho^{(0)} = \cdots = \rho^{(n)}$.

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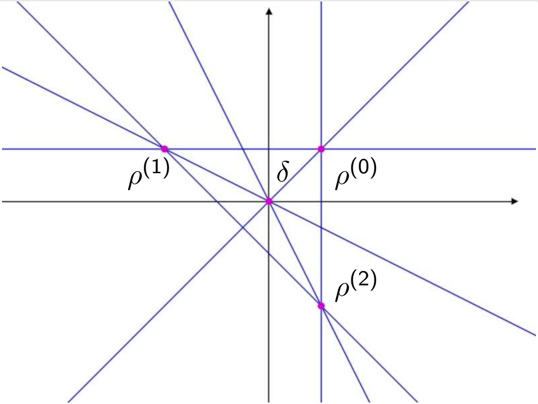

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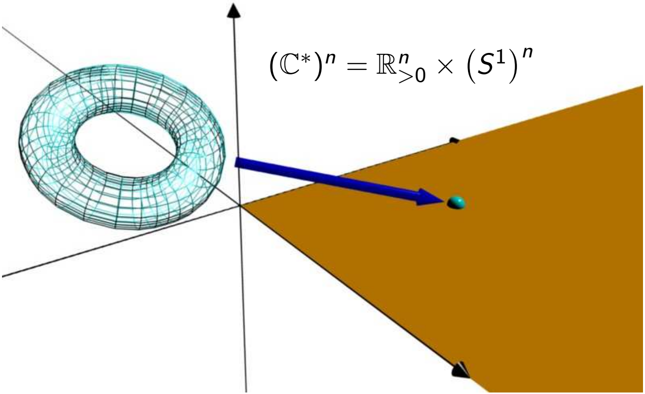
Idea:

Investigate the fibre $\mathrm{Log}^{-1}(\delta)$ of δ under the Log–map.

Sketch of proof

The fibre under the Log–map is an n–Torus

$$(\mathbb{C}^*)^n = \mathbb{R}^n_{>0} \times \left(S^1\right)^n$$



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This is part (b).

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$\Rightarrow \quad$ genus of $\mathcal{A}(f)$ is 1.

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Answer:

Only if

$$y = \frac{1}{n+1} \cdot \sum_{i=0}^{n} x^{(i)}$$

i.e. only if y is the barycenter of the simplex $\text{New}(f)$.

The position of y in $\mathrm{New}(f)$

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Obervation:

The inner complement component appears at the point $\delta + \nu$ of $\mathcal{A}(f)$ were — roughly spoken — the "inner monomial" has maximum weight with respect to the sum of all others.

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- In general nasty because $\delta + \nu$ and the boundary depends on the choice of $\arg(c)$.
- **BUT:** Everything becomes nice if **we** are allowed to choose $\arg(c)$ s.t. $|c|$ has to be maximal with respect to $\arg(c)$ until $\mathcal{A}(f)$ has genus 1.

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Theorem (Theobald, dW.)

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(b) For at last one $\arg(c)$ this boundary is sharp. For this $\arg(c)$ the hole will appear at the point $\nu + \delta \in \mathcal{A}(f)$ which can be computed explicitly.

Main Theorem

Theorem (Theobald, dW.)

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(c) For all $\arg(c)$ the exact boundary can be rewritten as the solution of some particular optimization problem.

Lopsidedness and A–discriminants

Lopsidedness

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It is easy to see that $\mathcal{A}(g) \subseteq \mathcal{LA}(g)$.

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$$\tilde{g}_r(\mathbf{z}) \quad := \quad \prod_{k_1=0}^{r-1} \cdots \prod_{k_d=0}^{r-1} g\left(e^{2\pi i k_1/r} z_1, \ldots, e^{2\pi i k_d/r} z_n\right)$$

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Theorem (Purbhoo 08)

For $r \to \infty$ the family $\mathcal{LA}(\tilde{g}_r)$ converges uniformly to $\mathcal{A}(g)$. $\mathcal{A}(g)$ can be approximated by $\mathcal{LA}(\tilde{g}_r)$ explicitly up to an $\varepsilon > 0$ if r is greater than some $N_{(\varepsilon,g)} \in \mathbb{N}$.

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Corollary (Theobald, dW.)

Let f be $(\Delta, \cdot)$. Then $\mathcal{A}(f)$ has genus 1 for all $\arg(c)$ if and only if $f\{\nu + \delta\}$ is lopsided with $\left|m_y\left(\mathrm{Log}^{-1}(\nu+\delta)\right)\right|$ as the maximal term.

The A–discriminant

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Proposition and definition

We define the A–DISCRIMINANT $\Delta_A(g)$ as an irreducible, integral polynomial in the configuration space $\mathbb{C}[\mathbf{b}]$ (with $\mathbf{b} := \prod_{x^{(i)} \in A} b_{x^{(i)}}$) vanishing exactly on ∇_A.

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(c) $\Delta_A(f)$ is given explicitly in $b_0, \ldots, b_n, c$ and the elements of A.

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Example

Let $f := 1 + b_1x^2y + b_2xy^2 + cxy$.

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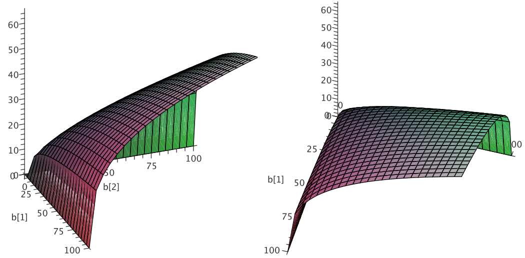

Thank you for your attention!