Assignment 1, Combinatorial Commutative Algebra, SS 12

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http://www.math.uni-sb.de/ag/schreyer/LEHRE/12_CombCom/index.htm

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In the following, we assume that $S = \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is an arbitrary field, and consider the standard $\mathbb{Z}$ grading for S.

Exercise 1.1 (**Complex of Vector Spaces**) Let

$$0 \to V_n \stackrel{\partial_n}{\to} V_{n-1} \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_1}{\to} V_0$$

be an exact complex of finite dimensional vector spaces over $\mathbb{K}$. Let β_i denote the dimension of V_i over $\mathbb{K}$, for every $0 \leq i \leq n$.

a) Show that

$$\mathsf{rank}_{\mathbb{K}}(\partial_i) = \beta_i - \beta_{i+1} + \beta_{i+2} - \ldots + (-1)^{n-i}\beta_n$$

b) As a corollary, if there is a zero on the right end i.e., for the exact sequence:

$$0 \to V_n \stackrel{\partial_n}{\to} V_{n-1} \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_1}{\to} V_0 \to 0$$

show that

$$\beta_0 = \beta_1 - \beta_2 + \ldots + (-1)^{n-1}\beta_n.$$

Exercise 1.2 Let M be a finitely generated module over S. Show that $\mathsf{Syz}_1(M)$ is also finitely generated over S. **Hint:** Hilbert's Basis Theorem.

Exercise 1.3 (**Graded Modules**) Let M and N be $\mathbb{Z}$-graded modules over S. Let $\varphi : M \to N$ be a degree preserving morphism of modules. Show that $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ are also graded S-modules, and hence, if we ensure that the differentials are degree preserving morphisms then the syzygy modules are graded.

Exercise 1.4 In this exercise, we will fill out the details of characterizations of all free resolutions of a module M in terms of its minimal free resolutions and trivial complexes.

Let $\mathcal{F}$ be a non minimal free resolution of the module M. By the lemma we proved in the lecture, one of the differentials ∂_i in $\mathcal{F}$ has a non-zero scalar leading to the trivial complex:

$$\mathcal{G} : 0 \to S(-a) \stackrel{\mathrm{id}}{\to} S(-a) \to 0$$

a) Show that

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F}/\mathcal{G} \to 0$$

is an short exact sequence of complexes.

b) From the above, deduce that the induced complex $\mathcal{F}/\mathcal{G}$ is the same as the complex $\mathcal{F}$ except that the induced differential ∂_i' does not have the corresponding scalar entry as in ∂_i.

Exercise 1.5 Let $I \subset S$ be an ideal. Show that the S-module S/I is graded by $\mathbb{Z}$ if and only if I is generated by homogeneous polynomials.