# Assignment 2, Combinatorial Commutative Algebra, SS 12

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http://www.math.uni-sb.de/ag/schreyer/LEHRE/12_CombCom/index.htm

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**Exercise 2.1** For any ideal $I$ of $S = \Bbbk[x_1, \ldots, x_n]$, show that the betti numbers of $I$ and $S/I$ are related by a shift in the homological degrees, i.e.,

$$\beta_i(I) = \beta_{i+1}(S/I)$$

for $i$ from 0 to $n-1$. **Hint:** Consider the short exact sequence $0 \to I \to S \to S/I \to 0$ and associated long exact sequence of Tor.

**Exercise 2.2** For each of the five Platonic solids (if unfamiliar with Platonic solids see Wikipedia entry on them), come up with a monomial ideal $I_P$ such that a labelling of the platonic solid $P$ resolves $I_P$. Furthermore, can you construct $I_P$ that is minimally resolved by a labelling of $P$?

**Exercise 2.3** Use your favourite computer algebra software (for example Macaulay 2) to compute the minimal free resolutions of $\mathsf{m}^2$ for various values of $n$, and conjecture on the betti numbers of $\mathsf{m}^2$. Can you prove your conjecture?

**Exercise 2.4** For a finite generated $\mathbb{Z}$-graded module over $S = \Bbbk[x_1, \ldots, x_n]$, define the $k$-th Hilbert coefficient $h_k = dim_{\Bbbk} M_k$ and the Hilbert Series $H(t) = \sum_{k \in \mathbb{Z}} h_k \cdot t^k$. Show that the $H(t)$ is a rational function i.e., there exist polynomials $f$ and $g$ such that $H(t) = f(t)/g(t)$.

**Hint:** Use the existence of the Hilbert polynomial of a finite generated graded module that we proved in the lecture.

**Exercise 2.5** Consider a finite generated $\mathbb{Z}$-graded module over $S = \Bbbk[x_1, \ldots, x_n]$. For any integer $j$, let $B_j = \sum_{i \geq 0} (-1)^i \beta_{i,j}$ where $\beta_{i,j}$ is the $i$-th Betti number of $M$ with twist $j$. Show that the Hilbert series $H(d)$ of $M$ is equal to $\sum_j B_j \binom{n+d-j}{n}$. Hence, note that the Graded minimal free resolution of $M$ has all the information needed to compute the Hilbert series of $M$.