Assignment 3, Combinatorial Commutative Algebra, SS 12

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http://www.math.uni-sb.de/ag/schreyer/LEHRE/12_CombCom/index.htm

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In this exercise sheet, we will study an application of Hilbert polynomial to counting the number of "magic squares". A *magic square* is an $n \times n$ matrix with entries over natural numbers is said to have a line sum r, if its row sums and column sums are equal to r. For example, when $r = 1$, the matrices with line sum 1 are exactly the set of $n \times n$ permutation matrices. Let $H_n(r)$ denote the number of $n \times n$ matrices with entries from $\mathbb{N}$ whose line sums are equal to r.

In this assignment, the goal is to show that for a fixed n, $H_n(r)$ is a polynomial in r, when r is large enough. For this purpose, we consider the positive integral solutions of a $\mathbb{Z}$-linear equation $A \cdot \mathsf{a} = \gamma$, where $A \in \mathbb{Z}^{k \times n}$, and $\mathsf{a} = (a_1, \ldots, a_n)$, $k \leq n$. Let R_A be the $\mathbb{K}$- subalgebra of $S = \mathbb{K}[x_1, \ldots, x_n]$ generated by $\{\mathsf{x}^\beta \mid \beta \in \mathbb{N}^n, A \cdot \beta = 0\}$ where $\mathsf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$. Hence, R_A has the natural $\mathbb{Z}^k$-grading, with $\mathsf{deg}(x^\beta) = \beta$. For any $\alpha \in \mathbb{Z}^k$, let $E_{A,\alpha} = \{\beta \in \mathbb{N}^n \mid A \cdot \beta = \alpha\}$, and $M_{A,\alpha}$ be the $\mathbb{K}$-linear span of $\{\mathsf{x}^\beta \mid \beta \in E_{A,\alpha}\}$.

Exercise 3.1 Show that $M_{A,\alpha}$ is a $\mathbb{Z}^n$-graded and in particular $\mathbb{Z}$-graded R_A-module, with the grading $\mathsf{deg}(x^\beta) = \beta$.

Exercise 3.2 Show that R_A is a finitely generated $\mathbb{K}$-algebra. **Hint:** Consider the S-ideal I generated by $\{x^\beta \mid \beta \in \mathbb{N}^n, A \cdot \beta = 0\}$ and then show that a generating set for I also generates R_A as a $\mathbb{K}$-algebra.

Exercise 3.3 Show that $M_{A,\alpha}$ is a finitely generated module over R_A.

Exercise 3.4 Let n be fixed. Construct a suitable system of $\mathbb{Z}$-linear equations $A \cdot \mathsf{a} = \gamma$, and consider the corresponding Hilbert series of the module $M_{A,\gamma}$ under the $\mathbb{Z}$-grading. Deduce that $H_n(r)$ is a polynomial in r, where r is large enough.

Remark 1 *The conjecture of $H_n(r)$ being a polynomial for a fixed value of n was made by Anand, Dumir and Gupta in 1966. We saw that $H_n(r)$ is a polynomial for large enough r. In fact, $H_n(r)$ is a polynomial for all values of r. See Chapter I of Stanley's book* Combinatorics and Commutative Algebra *for a full proof of the conjecture and a more detailed study of positive integer solutions of $\mathbb{Z}$-linear equations.*

Remark 2 *As we shall see in the lecture, the notion of a grading can be defined (not just over $\mathbb{Z}$ or $\mathbb{Z}^n$), but over an additively closed set G, i.e semi-groups or monoids. However, for the purpose of defining twists, G needs to have additive inverses.*