Assignment 4, Combinatorial Commutative Algebra, SS 12

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http://www.math.uni-sb.de/ag/schreyer/LEHRE/12_CombCom/index.htm

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Let $S = \mathbb{K}[x_1, \ldots, x_n]$. For a monomial ideal $M = \langle m_1, \ldots, m_t \rangle$, let $\mathcal{C}(M)$ be the simplicial complex with $\{m_1, \ldots, m_t\}$ as vertices. Let all subsets $F \subseteq \{m_1, \ldots, m_t\}$ with $LCM(m \mid m \in F)$ not strictly divisible [1] by any of the monomials in the minimal generating set, and all subsets of F be the faces of $\mathcal{C}(M)$. Note that $\mathcal{C}(M)$ is a labelled simplicial complex with a natural labeling, i.e., vertex m_i is labelled with the monomial m_i. The objective of this exercise is to show

Theorem 1 *The chain complex of $\mathcal{C}(M)$ with the labeling as above is an exact complex of free modules with $\mathsf{coker}(\partial_1) = S/M$, i.e., $\mathcal{C}(M)$ is a free resolution of S/M.*

Let $B = \{m_1, \ldots, m_t\}$. Using the acyclity criterion for exactness we know that $\mathcal{C}(M)$ is exact if and only if $(\mathcal{C}(M))_{\leq(b_1,\ldots,b_n)}$ is acyclic for every $(b_1, \ldots, b_n) \in \mathbb{N}^n$. In fact, in the following exercises, we will show that $(\mathcal{C}(M))_{\leq(b_1,\ldots,b_n)}$ is not only acyclic but also contractible.

Exercise 4.1 Show that $\mathcal{C}(M)_{\leq(b_1,\ldots,b_n)}$ is homotopy equivalent to

$$\bigcup_{m\in B, m|x_1^{b_1}\cdots x_n^{b_n}} \partial(H^+(m))$$

where $H^+(m)$ is the translate of the positive orthant cone with apex at $(a_1, \ldots, a_n)$ where $m = x_1^{a_1} \cdots x_n^{a_n}$.

Exercise 4.2 Show that $\bigcup_{m\in B, m|x_1^{b_1}\cdots x_n^{b_n}} \partial(H^+(m))$ is homeomorphic to a disc. **Hint** Embed the exponents of M ring in sufficiently high dimension such that the drawing of the simplicial complex $\mathcal{C}(M)$ has no self intersections. Then lift a point $p \in \mathcal{C}(M)_{\leq(b_1,\ldots,b_n)}$ along $(1, \ldots, 1)$ till it intersects $\bigcup_{m\in B, m|x_1^{b_1}\cdots x_n^{b_n}} \partial(H^+(m))$.

Exercise 4.3 Construct an example where $\mathcal{C}(M)$ has dimension equal to the size of the set of minimal generators of M.

[1] a monomial m' strictly divides m, if $m'|m$ and degree each of the variable occurring in m' is strictly smaller than that in m.